On Septic Scrolls Having a Rectilinear Directrix.

By Charles H. Sisam.

Introduction.

Methods of Classifying Ruled Surfaces.

1. The most obvious method of classifying ruled surfaces is by means of their (x, y, z, w) equations. This method was successfully used by Salmon* in his classification of cubics and quartics. For surfaces of higher degree, however, this method is too cumbersome to be employed.

2. Salmon † also considered ruled surfaces geometrically as the loci of lines which cut three fixed curves on them. Many ruled surfaces are not, however, the complete locus of lines cutting three curves on them; and this method gives very little information concerning the separate components when the locus is composite.

3. An important variation of the above is obtained by considering the locus of a line which cuts one curve twice and another once. Such surfaces will frequently be referred to in this article as "scrolls of bisecants," the second curve, however, always being straight line. The degree of such a scroll of bisecants is,‡ in general, $h - m'\frac{(m'-1)}{2} + 1/2\,(m-m')\,(m-m'-1)$ where m is the order of the curve, m' the number of points in which the curve meets the straight line, and h the number of its apparent double points. The straight line is an $h - \frac{m'(m'-1)}{2}$-fold line and the curve an $(m-m'-1)$-fold curve on the surface.§

4. A further variation I shall occasionally refer to is the locus of a line cutting a curve thrice. The degree of such a surface is|| $(m-2)\;[h - 1/6m\,(m-1)]$.

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* Geometry of three dimensions, 4th Ed., pp. 485–8, and 512–22.

† On a class of ruled surfaces, Cambridge and Dublin Math. Journal, Vol. VIII, p. 45.

‡ Salmon, Geometry of three dimensions, 4th Ed., p. 431.

§ The condition that the surface be composite will be obtained later.

|| Salmon, i. b., i. d., p. 432.

5. Schwarz,* in his excellent classification of quintic scrolls, considers them as the loci of the lines of intersection of corresponding planes of two developables between which a one to one correspondence has been established. This method is not, however, usually so useful as the dual of it which has been applied by Fink† and by Snyder‡ to the classification of sextics. Consider two simple curves on a ruled surface such that each is cut by an arbitrary generator only once. To each point of one curve corresponds one point of the other lying on the same generator. Conversely, if a one to one correspondence is set up between the points of two curves, then the locus of the lines joining corresponding points is a ruled surface. This method has the advantage that it leads at once to the parametric equations of the surface.

6. Ruled surfaces known to belong to a given line complex may be studied by means of a curve theory if we apply a contact transformation which transforms the lines of the complex into the points of space. This is the method used by Wiman § in his classification of sextics and is the one which will chiefly be followed in the present investigation.

Notation and Theorems.

7. I shall use the following notation:

R_n = Scroll of degree n.

p = genus of a scroll.

P_i = point, i-fold on the surface.

g_i = i-fold generator.

$(2g_2)$ = Double torsal generator.|| See Par. 66.

$(3g_2)$ = A double torsal and a consecutive double generator. See Par. 67.

d_i = i-fold directrix.

$(d_i + jg_1)$ = i-fold directrix with which j generators coincide.

$(d + jg_1 + kg_2)$ = i-fold directrix with which j simple and k double generators coincide.

$(\delta_{k,l} + jg_1)$ = kl-fold contact directrix, *i. e.* one such that of the kl generators passing through a point, k lie in each of l planes through the directrix. In addition j generators coincide with the directrix.

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* Schwarz, Über die gradlinigen Flächen fünften Grades," Crelle's Journal, Vol. 67.

† Fink, "Über windschiefe Flächen," etc. Diss., Tübingen, 1886.

‡ Snyder, "Classification of Sextic Scrolls," etc. *American Journal of Mathematics*, Vol. XXV, pp. 59–84, 85–96, 261–268. Vol. XXVII, pp. 77–102, 173–188.

§ Wiman, "Regelytorna af Sjette Graden," Diss., Lund, 1892.

|| Wiman's "Singulär dubbelgeneratrix," i. b. i. d., p. 27.

K_n = cone of order n.

C_m^i = curve of order m which is i-fold on the surface.

p' = genus of a curve.

P_i' = point, i-fold on a curve.

8. The point of highest multiplicity that exists in general on a ruled surface is a P_3. If we consider a P_i equivalent to $\frac{i(i-1)(i-2)}{6} P_3$, then the number t, of P_3 on an R_n having a $(d_i + jg)$ is* $t = 1/6\,(n-i-j-2)[(n-i-j-1)(n+2i+2j-6)-6p]$.

9. The maximum genus of the nodal curve is

$$p' = 1/2\,(n-i-j-2)(n-i-j-3) + p\,(n-i-j-2)\dagger$$

10. Denoting by m' the number of intersections of a C_m^a with a $(d_i + jg)$ of an R_n and by b the number of intersections of the curve with an arbitrary generator, then

$$a\,(m - m') = b\,(n - i - j),$$

since each member of this equation equals the number of intersections of the curve with the $n - i - j$ generators in an arbitrary plane through the directrix.

11. It is also readily seen that for the entire nodal curve (including the directrix and multiple generators)

$$\sum \frac{ma\,(a-1)}{2} = \frac{(n-1)(n-2)}{2} - p$$

$$\sum (a-1)\,b = n-2$$

CLASSIFICATION OF SEPTIC SCROLLS HAVING A RECTILINEAR DIRECTRIX.

I. Scrolls Belonging to Linear Congruences.

a. Scrolls Belonging to General Linear Congruences.

12. The sum of the multiplicities i_1 and i_2 of the two rectilinear directrices of the scroll must equal n, the degree of the surface. Hence if we take $x = y = 0$ to be one directrix and $z = w = 0$ to be the other, then the equation of the surface must be homogeneous of degree i_1 in x and y, and of degree i_2 in z and w. Putting $\frac{x}{y} = \xi$, $\frac{z}{w} = \eta$ and regarding ξ, η as rectangular coordinates in a plane, we obtain the equation of a curve which is the transform, in the ξ, η plane, of

* Wiman, loc. cit., p. 10.

† Wiman, loc. cit., p. 12.

the given surface.* It is, in general, of order n and has an i_1-fold point at infinity on the $\xi = 0$ axis and an i_2-fold point at infinity on the $\eta = 0$ axis. To the remaining multiple points of the curve correspond multiple generators of the scroll. It is, in fact, projective with the section of the surface by $y = w$.

13. For $n = 7$ we may take for i_1 and i_2 the values 1 and 6, 2 and 5, or 3 and 4. We thus obtain the following R_7 having two rectilinear directrices.

$p = 0$

1. $d_1 + d_6$
2. $d_2 + d_5 + 4g_2$
3. $d_3 + d_4 + 6g_2$
4. $d_3 + d_4 + g_3 + 3g_2$
5. $d_3 + d_4 + 2g_3$

$p = 1$

6. $d_2 + d_5 + 3g_2$
7. $d_3 + d_4 + 5g_2$
8. $d_3 + d_4 + g_3 + 2g_2$

$p = 2$

9. $d_2 + d_5 + 2g_2$
10. $d_3 + d_4 + 4g_2$
11. $d_3 + d_4 + g_3 + g_2$

$p = 3$

12. $d_2 + d_5 + g_2$
13. $d_3 + d_4 + 3g_2$
14. $d_3 + d_4 + g_3$

$p = 4$

15. $d_2 + d_5$
16. $d_3 + d_4 + 2g_2$

$p = 5$

17. $d_3 + d_4 + g_2$

$p = 6$

18. $d_3 + d_4$

14. Particular cases of consecutive, tacnodal, oscnodal, etc., multiple generators may be obtained from corresponding particular cases of the transformed curve. I shall not usually call attention to these obvious particularizations.

b. Scrolls Belonging to Special Linear Congruences.

15. The equations of any special linear congruence may, by a suitable choice of coordinates, be written $p_{12} = 0$, $p_{14} = p_{23}$. The equations of a scroll belonging to this congruence are:

$$x = a(u) \qquad z = c(u) + va(u)$$
$$y = b(u) \qquad w = d(u) + vb(u)$$

from which $\dfrac{x}{y} = \dfrac{a}{b}$; $\dfrac{xw - yz}{y^2} = \dfrac{ad - bc}{b^2}$. Hence the (x, y, z, w) equation of

* For this transformation I am indebted to Professor Snyder.

the surface is* $\sum_0^i{}_r (xw - yz)^r \cdot q_{n-2r}(x, y)$ where n is the order of the surface, $i \leqq \frac{n}{2}$ and $q_{n-2r}(x, y)$ is a binary quantic in x and y of degree $n - 2r$.

16. Putting $\frac{x}{y} = \xi$ and $\frac{xw - yz}{y^2} = \eta$ we obtain, as transform of the surface, a curve projective with the section of the surface by $y = w$. This curve is, in general, of order n. It has an $(n - i)$-fold and a consecutive i-fold point at infinity on the $\xi = 0$ axis. If, however, this point counts for more than $\frac{(n-i)(n-i-1)}{2} + \frac{i(i-1)}{2}$ double points, then multiple generators will coincide with the directrix. For $n = 7$, $i = 1, 2, 3$ we obtain:

	$p = 0$		$p = 2$
1.	$(\delta_{1,1} + 5g_1)$	13.	$(\delta_{3,1} + g_1) + 4g_2$
2.	$(\delta_{2,1} + 3g_1) + 4g_2$	14.	$(\delta_{3,1} + g_1) + g_3 + g_2$
3.	$(\delta_{2,1} + g_1 + g_2) + 3g_2$		$p = 3$
4.	$(\delta_{3,1} + g_1) + 6g_2$	15.	$(\delta_{2,1} + 3g_1) + g_2$
5.	$(\delta_{3,1} + g_1) + g_3 + 3g_2$	16.	$(\delta_{2,1} + g_1 + g_2)$
6.	$(\delta_{3,1} + g_1) + 2g_3$	17.	$(\delta_{3,1} + g_1) + 3g_2$
	$p = 1$	18.	$(\delta_{3,1} + g_1) + g_3$
7.	$(\delta_{2,1} + 3g_1) + 3g_2$		$p = 4$
8.	$(\delta_{2,1} + g_1 + g_2) + 2g_2$	19.	$(\delta_{2,1} + 3g_1)$
9.	$(\delta_{3,1} + g_1) + 5g_2$	20.	$(\delta_{3,1} + g_1) + 2g_2$
10.	$(\delta_{3,1} + g_1) + g_3 + 2g_2$		$p = 5$
	$p = 2$	21.	$(\delta_{3,1} + g_1) + g_2$
11.	$(\delta_{2,1} + 3g_1) + 2g_2$		$p = 6$
12.	$(\delta_{2,1} + g_1 + g_2) + g_2$	22.	$(\delta_{3,1} + g_1)$

II. Scrolls not Belonging to Linear Congruences.

Directrix a Simple Line on the Surface.

17. The nodal curve is limited very closely by the relations given in paragraphs 10 and 11. For the present case these reduce to $ma = 6b$, $\sum m \frac{a(a-1)}{2} = 15$, $\Sigma (a - 1) b = 5$. If $a = 6$, $m = 1$ and the R_7 belongs to a linear congruence, a can not equal 5 or 4. If $a = 3$ then $m = 2b$ hence every triple curve is of even order. Similarly, for a double curve, $m = 3b$.

* Snyder, Bulletin Amer. Math. Soc. Vol. V, p. 351.

18. Since the surface is unicursal its parametric equations are:

$$x = a_6(u), \quad z = c_6(u) + v(\alpha u + \beta)$$
$$y = b_6(u), \quad w = d_6(u) + v(\gamma u + \delta)$$

where $a_6(u)$, $b_6(u)$, $c_6(u)$ and $d_6(u)$ are polynomials of sixth degree in u and $a_6(u)$ and $b_6(u)$ are relative prime.

19. If we form the function*

$$F_5(uu', u + u') = \frac{a_6(u)\, b_6(u') - b_6(u)\, a_6(u')}{u - u'},$$

then the number of components of the nodal curve and the genus of the components which are double curves are determined by the corresponding properties of $F_5 = 0$ considered as a curve whose current coordinates are $u u'$ and $u + u'$.

20. In the present case, the order of each double component of the nodal curve is thrice, and of each triple component is once, the order of the corresponding component of $F_5 = 0$.

21. The curve $F_5 = 0$ exhibits the following forms:

1. a proper C_5 of genus 6, 5, 4, 3, 2, 1, 0.
2. a C_2 and a C_4 of genus 3, 2, 1, 0.
3. a C_2 and a C_3 of genus 1, 0.
4. a C_1 and $2C_2$.
5. $3C_1$ and a C_2.

The forms $5C_1$ and $2C_1 + C_3$ do not exist provided $a_6\ (u)$ and $b_6\ (u)$ are relative prime.

a. Triple Curves.

22. A triple curve lying on an R_7 with a d_1 must be either a C_4^3 or a C_2^3.

To a C_4^3 on R_7 corresponds a quartic component of $F_5 = 0$. To the remaining linear component of $F_5 = 0$ corresponds a C_3^2 on R_7. This surface is readily obtained by Salmon's geometrical method (see p. 1). The scroll of bisecants from a unicursal C_4 to a straight line is, in general, an R_9 having the line for d_3. If however, the C_4 has a P_2' at which both tangents cut the straight line, then the scroll reduces to an R_7 having the line for d_1 since the P_2' will project from any point of the line into a tacnode. The R_7 has a P_4 at the P_2' of the C_4^3. The residual C_3^2 on the R_7 passes simply through that point. It also passes through

* Cf. my article in this Journal, Vol. XXVIII, p. 43

each of the remaining points at which the tangent to the C_4^3 cuts d_1. The tangent to C_3^2 at each of these points also cuts d_1, since the six generators in the plane through the point and d_1 have 8 intersections at the point.

23. To a C_2^3 on the R_7 corresponds a quadratic component of $F_5 = 0$. The R_7 can not have $2C_2^3$ for the three generators through a point of one C_2 would have to cut the other C_2 in two points in the plane through the given point and the directrix. The residual nodal curve is, therefore, C_9^2 or $C_6^2 + C_3^2$ or $3C_3^2$. The R_7 has a P_4 on the C_2^3 for the plane of the C_2 contains a g which cuts the C_2 in two points, fourfold in the curve of section. One is the point at which the plane touches the surface, the other is a P_4 on the surface. The residual nodal curve must have a P_3' at the P_4 since the total nodal curve must have a P_6' there. The residual curve also has a P_3' at each point from which a tangent to C_2^3 cuts the directrix, for the six generators in the plane through the directrix and either of these points must consist of 3 torsals meeting at the point of tangency. When the double curve has a component C_3^2, two of the points at which the tangent to the C_3^2 cuts the directrix are these two points, the other two are points at which the C_3 cuts the remaining double curve.

24. We have, therefore, the following types:

1. $d_1 + C_4^3 + C_3^2$. R_7 has a P_4 at the P_2' of the C_4. The C_3 is gauche. It passes through the P_4 and cuts C_4 in four other points at which the tangents to both curves cut d_1.

2. $d_1 + C_2^3 + C_9^2$. The C_9 has a P_3' at the P_4 and a P_3' at each point of the C_2^3 at which the tangent to C_2^3 cuts d_1. At each of these points the three tangents to C_9 cut d_1. p' for C_9 is in general 1, but may reduce to o by the appearance of a P_2' at which both tangents cut d_1.*

3. $d_1 + C_2^3 + C_6^2 + C_3^2$. The C_3 goes through the P_4 and the points at which the tangents to C_2 cut d_1. The C_6 has a P_2' at each of these three points and cuts the C_3 in two other points.

4. $d_1 + C_2^3 + 3C_3^2$. Each C_3 goes through the P_4 and the points from which the tangents to C_2 cut d_1. Each cuts each of the other C_3 again.

b. No Triple Curves.

25. The scroll determined by equations I, paragraph 18, may be made, by changes in the expressions for z and w only, to have for its point of highest multi-

*In general, both tangents at a discrete P'_2 on a double curve must cut the directrix. Such points are the intersections of torsal generators having a common torsal plane.

plicity a P_5, a P_4 or a P_3 hence among each of these three kinds of scrolls are types determined by 1, 2, 3, 4, 5. No. 34.

Fivefold Point.

26. The double curve has a P'_{10} at the P_5 and therefore lies on a K_5. The curve $F_5 = 0$ considered above is projective with an arbitrary plane section of this K_5. Each C^2_m has a point of multiplicity $\frac{2}{3} m$ at the P_5. On each multiple generator of the K_5 lies P'_2 of the double curve. Hence we have:

1. $d_1 + C^2_{15}$. The C_{15} has a P'_{10} at the P_5. $p' = 6, 5, 4, 3, 2, 1, 0$ according as it has 0, 1, 2, 3, 4, 5, 6, P'_2.

2. $d_1 + C^2_3 + C^2_{12}$. The C_3 has a P'_2 and the C_{12} a P'_8 at P_5. They meet in four other points. p' for the C_{12} is 3, 2, 1, 0 according as it has 0, 1, 2, 3, P'_2.

3. $d_1 + C^2_6 + C^2_9$. The C_6 has a P'_4 and the C_9 a P'_6 at P_5. They meet in six other points. p' for C^2_9 is 1 or 0.

4. $d_1 + C^2_3 + 2C^2_6$. The C_3 has a P'_2 and each C_6 a P'_4 at P_5. The C_3 meets each C_6 in $2P'_1$ and the $2C_6$ meet in $4P'_1$.*

5. $d_1 + 3C^2_3 + C^2_6$. Each C_3 has a P'_2 and the C_6 a P'_4 at P_5. Each C_3 meets each of the other C_3 once and the C_6 twice.

Fourfold Point.

27. The double curve has a P'_3 at each of the $6P_3$ and a P'_6 at the P_4. In the plane through the P_4 and the d_1 lie $2g_1$ not passing through the P_4. When the double curve is composite, there are different types according as the point of intersection of these generators lies on one or another component.† The only apparent double points of the nodal curve from the P_4 are the four in the plane through the P_4 and d_1.

1. $d_1 + C^2_{15}$. The C_{15} has a P'_6 at the P_4 and a P'_3 at each of the $6P_3$. p' equals 6, 5, 4, 3, 2, 1, or 0 according to the number of its P'_2.

2. $d_1 + C^2_3 + C^2_{12}$. The C_3 passes through the P_4 and $4P_3$. The C_{12} has a P'_5 at the P_4, P'_3 at $2P_3$ and P'_2 at $4P_3$. It meets C_3 in $4P'_1$. p' for C_{12} is 3, 2, 1, 0.

* The six generators in an arbitrary plane through the directrix are the sides of a Pascal hexagon. The vertices of the hexagon are the six points in the plane on one C^2_6, the conic on which they lie being the section, by the plane, of the K_2 with vertex at P_4 which contains this C^2_6. The vertices of the Steiner triangles are on the other C^2_6 and the points on the Pascal line are on the C^2_3.

† The existence of these distinct cases may be verified readily by constructing the R_7 so that the parameters of these generators satisfy first one and then another component of $F_5 = 0$.

3. $d_1 + C_3^2 + C_{12}^2$. The C_3 has a P_2' at P_4 and passes through $2P_3$. The C_{12} has a P_4' at P_4, P_3' at $4P_3$ and P_2' at $2P_3$. It meets C_3 in $4P_1'$. p' for C_{12} is 3, 2, 1 or 0.

4. $d_1 + C_6^2 + C_9^2$. The C_6 has a P_3' at the P_4 and a P_1' at each P_3. The C_9 has a P_3' at the P_4 and a P_2' at each P_3. It meets C_6 in $6P'$. p' for C_9 is 1 or 0.

5. $d_1 + C_6^2 + C_9^2$. The C_6 has a P_2' at the P_4, a P_3' at one P_3 and P_1' at the remaining P_3. The C_9 has a P_4' at the P_4 and P_2' at $5P_3$. It meets C_6 in $6P_1'$. p' for C_9 is 1 or 0.

6. $d_1 + C_3^2 + 2C_6^2$. The C_3 passes through P_4 and through $4P_3$ and meets each C_6 in two other points. One C_6 has a P_3' at the P_4, a P_2' at one P_3, one P_1' at $4P_3$. The other C_6 has a P_2' at the P_4, a P_3' at one P_3 and P_1' at $5P_3$. It meets the other C_6 in $4P_1'$.

7. $d_1 + C_3^2 + 2C_6^2$. The C_3 as in type 6. One C_6 has a P_3' at the P_4 and P_1' at $6P_3$. The other C_6 has P_2' at the P_4 and at $2P_3'$. It has P_1' at the other $4P_3$ and meets the first C_6 in four other P_1'.

8. $d_1 + C_3^2 + 2C_6^2$. The C_3 has a P_2' at the P_4, P_1' at $2P_3$ and cuts each C_6 in two other P_1'. Each C_6 has a P_2' at the P_4. One C_6 has a P_3' at one P_3 and P_1' at $5P_3$. The other C_6 has P_2' at $3P_3$ and P_1' at $2P_3$. They meet in four other P_1'.

9. $d_1 + 3C_3^2 + C_6^2$. Each C_3 has P_1' at the P_4 and at $4P_3$. Each meets the C_6 in $2P_1'$ and each of the other C_3 in $1P_1'$. The C_6 has a P_3' at the P_4 and a P_1' at each P_3.

10. $d_1 + 3C_3^2 + C_6^2$. One C_3 has a P_2' at P_4 and P_1' at $2P_3$. The other $2C_3$ are as in type 9. The C_6 has a P_2' at P_4, a P_3' at one P_3 and P_1' at the other $5P_3$.

Ten Threefold Points.

28. The R_7 cannot have a plane C_3^2 for the P_2' on the C_3 would be at least a P_4 on the R_7. When any component of the nodal curve has a P_1' at a P_3 then one generator through that point cuts it in one more point than an arbitrary generator does and conversely. Hence every C_3^2 has P_1' at $6P_3$ since the R_7 has $6g_1$ in common with the R_4 of bisecants from the directrix to C_3. Similarly no C_6^2 can lie on a quadric for the total intersection of the quadric and R_7 would be of higher degree than 14.

1. $d_1 + C_{15}^2$. The C_{15} has a P_3' at each P_3. $p' = 6, 5, 4, 3, 2, 1$ or 0.

2. $d_1 + C_3^2 + C_{12}^2$. The C_3 has P_1' at $6P_3$. The C_{12} has P_2' at these $6P_3$ and P_3' at the other $4P_3$. It meets C_3 in $4P_1'$. p' for C_{12} is 3, 2, 1, or 0.

3. $d_1 + C_6^2 + C_9^2$. The C_6 has a P_3' at $1P_3$ and a P_1' at each of the other $9P_3$. The C_9 has P_2' at these $9P_3$ and meets the C_6 in six other P_1'. p' for C_9 is 1 or 0.

4. $d_1 + C_3^2 + 2C_6^2$. The C_3 goes through $6P_3$ and meets each C_6 in two other P_1'. One C_6 has a P_2' at one P_3 and a P_1' at each of the other $9P_3$. The other C_6 has P_2' at $3P_3$ and P_1' at six other P_3. It meets the other C_6 in four other P_1'.

5. $d_1 + 3C_3^2 + C_6^2$. Each C_3 goes through $6P_3$ and meets each of the other C_3 in one other P_1' and the C_6 in two other P_1'. The C_6 has a P_3' at $1P_3$ and a P_1' at each of the other $9P_3$.

Digression on a Point-Line Contact Transformation.

29. Of the infinite number of transformations that transform the lines of a special linear complex into the points of space, only one need be considered. For any such transformation is equivalent to a projection, followed by the transformation mentioned below, and that followed by a point transformation. This transformation has already been fully discussed by Wiman.*

30. The polar planes for a point P with respect to a pencil of quadric surfaces form a pencil of planes having a line L for axis. Let the pencil of quadrics be so chosen that one member consists of two planes intersecting in a line M. Then the lines L determined by all the points of space cut M. A one to one correspondence is thus established between the points of space and the lines of a special linear complex whose axis is M.

31. To the points of a line of the complex correspond, by the theory of pole and polar, the lines of the complex through the corresponding point. To an R belonging to the complex corresponds a C in space. To any curve C' on R corresponds a ruled surface containing C. In particular, to the double curve of R corresponds the ruled surface formed by the complex lines which cut C twice.

Two successive applications of the transformation obviously produce identity.

32. Let the equations of the pencil of quadrics be:

$$x^2 + y^2 + \lambda\,(y^2 + z^2 + w^2) = 0.$$

To the point $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$, then, corresponds the line determined by any two of the equations:

$$x\bar{x} + y\bar{y} = 0 \tag{1}$$

$$y\bar{y} + z\bar{z} + w\bar{w} = 0 \tag{2}$$

$$x\bar{x} - z\bar{z} - w\bar{w} = 0 \tag{3}$$

* Regelytorna of Sjette Graden, Lund, Diss., 1892, p. 17.

33. The R corresponding to the intersection of two surfaces $\phi_1(\bar{x}, \bar{y}, \bar{z}, \bar{w})=0$ and $\phi_2(\bar{x}, \bar{y}, \bar{z}, \bar{w})=0$ is found by eleminating $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ between $\phi_1=0$, $\phi_2=0$ and any two of equations (1), (2), and (3). The surface corresponding to the curve $\bar{x}=f_1(u)$, $\bar{y}=f_2(u)$, $\bar{z}=f_3(u)$, $\bar{w}=f_4(u)$ is found by substituting these values in (1), (2) and (3) and eliminating u. The curve corresponding to a given ruled surface is found in a similar manner.

34. The fundamental points of the correspondence, *i. e.*, those which make any two of equations (1), (2) and (3) identical, are $\bar{x}=\bar{z}=\bar{w}=0$, $\bar{y}=\bar{z}=\bar{w}=0$ and the points of $\bar{x}=\bar{y}=0$. The order of the scroll corresponding to given curve is obviously diminished by unity for every time the curve goes through a fundamental point.

35. For convenience the points $\bar{x}=\bar{z}=\bar{w}=0$ and $\bar{y}=\bar{z}=\bar{w}=0$ only will be referred to as fundamental points. To the point $\bar{x}=\bar{z}=\bar{w}=0$ on a curve corresponds, obviously, a line in $y=0$. If the direction of the curve at this point is given by $d\bar{x}$, $d\bar{z}$, $d\bar{w}$ then the line in $x=0$ is determined by $xd\bar{x}-zd\bar{z}-wd\bar{w}$. Similarly for curves through $\bar{y}=\bar{z}=\bar{w}$. To the ∞^2 lines in $x=0$ and $y=0$ thus correspond the ∞^2 directions through the fundamental points.

36. By equations (1) and (3), it is seen that to all the points in $\bar{x}=0$ on a line through $\bar{x}=\bar{z}=0$ (except $\bar{x}=\bar{z}=\bar{w}=0$ itself and the point in $\bar{y}=0$) correspond the same line $y=0$, $z\bar{z}+w\bar{w}=0$. Multiple generators on R are thus accounted for either by multiple points on C or by points on lines of the complex through the fundamental points, according as the generator does not or does pass through a singular point.

37. Let the tangent to C at a point of intersection with $\bar{x}=\bar{y}=0$ lie in the plane $\bar{x}d\bar{y}-\bar{y}d\bar{x}=0$. The corresponding line is $xd\bar{x}+yd\bar{y}+0$, $z\bar{z}+w\bar{w}=0$. It obviously cuts $z=w=0$. The number of intersections of R with $z=w=0$ at points other than fundamental points equals the number of intersections of C with $\bar{x}=\bar{y}=0$.

38. To the points of C in any plane $z+\lambda\bar{w}=0$ (except the fundamental points) correspond generators of R through $(0, 0, 1, \lambda)$. To the points common to all the planes correspond lines through all the points; *i. e.*, the directrix counts as many times as a generator as C intersects $\bar{z}=\bar{w}=0$ in points other than fundamental points.

39. To a C_m cutting $\bar{x}=\bar{y}=0$ α times, having a P'_β at $\bar{x}=\bar{z}=\bar{w}=0$ and a P'_γ at $\bar{y}=\bar{z}=\bar{w}=0$ corresponds an $R_{2m-\alpha-\beta-\gamma}$ having $x=y=0$ for an

$(m-\beta-\gamma)$-fold line, having a $P_{m-\alpha-\beta}$ at $y=z=w=0$ and a $P_{m-\alpha-\gamma}$ at $x=z=w=0$. Similarly to an R_n having $x=y=0$ for α-fold line, having a P_β at $x=z=w=0$ and a P_γ at $y=z=w=0$ corresponds a $C_{2m-\alpha-\beta-\gamma}$ cutting $\bar{x}=\bar{y}=0$, $n-\beta-\gamma$ times, having a $P'_{n-\alpha-\beta}$ at $\bar{y}=\bar{z}=\bar{w}=0$ and a $P'_{n-\alpha-\gamma}$ at $\bar{x}=\bar{z}=\bar{w}=0$.

40. To the pencil of complex lines in an arbitrary plane corresponds a conic, cutting $\bar{x}=\bar{y}=0$ and passing through both fundamental points. To the points of the plane correspond the complex lines cutting the conic. We may, therefore, consider this conic as the transform of the plane. Since each tangent plane to R contains a generator, the corresponding conic cuts C. To the planes of the double developable of R (*i. e.*, the developable enveloped by planes containing two generators of R) correspond the conics which cut C twice. But these conics are in one to one correspondence with the lines of the complex $p_{34}=0$ which cut C twice. Hence the double developable is in one to one correspondence with the scroll of bisecants from $\bar{z}=\bar{w}=0$ to C.

41. To the points of an arbitrary straight line corresponds a quadric surface containing $x=y=0$ and the fundamental points. To an R of the complex which has this line for a second rectilinear directrix corresponds, therefore, a C lying on this quadric surface. In particular when the quadric contains $z=w=0$ the corresponding line is $\bar{x}=\bar{y}=0$. To a C lying on such an R_2 corresponds an R belonging to a special linear congruence.

42. To any line passing through a fundamental point corresponds a plane pencil passing through the other fundamental point. To an R of the complex having this line for a second directrix corresponds a C in the plane of the corresponding pencil.

43. To a point of C at which the tangent cuts $\bar{x}=\bar{y}=0$ corresponds a torsal generator of R with its pinch-point not on the directrix. To a point of C at which the tangent cuts $\bar{z}=\bar{w}=0$ corresponds a torsal with its pinch-point on the directrix.

44. It is occasionally more convenient to use another transformation. A brief mention of some of its properties is therefore appended. It is projectively equivalent to Euler's contact transformation.* It was also discussed by Wiman.† When it is necessary to distinguish between these two transformations the one above will be referred to as transformation I and the following as transformation II.

* Lie-Scheffers Berührungstransformationen, 1896, Vol. I, p. 647.

† Wiman, loc. cit. p. 23.

45. Let the equation of the pencil of quadrics be:

$$2xy + (2yw + z^2) = 0$$

The transformation is determined by the equations:

$$x\bar{y} + y\bar{x} = 0$$
$$y\bar{w} + z\bar{z} + w\bar{y} = 0$$

46. The singular points are those on $x = y = 0$ and on $z = w = 0$. The degree of the surface corresponding to C is decreased by unity for each of its intersections with either of these lines. If C passes through the point of intersection of the lines, the degree is reduced only by unity unless C touches $\bar{y} = 0$ at the point, in which case the degree is reduced by two.

47. To points of C in $\bar{y} = 0$ not on the fundamental lines correspond generators coincident with $y = z = 0$. To points on $\bar{y} = \bar{z} = 0$ correspond generators in $y = 0$ and to points on $\bar{x} = \bar{y} = 0$ correspond generators through $x = y = z = 0$ not, in general, in $y = 0$.

48. Hence to a C_n having α points on $\bar{x} = \bar{y} = 0$ and β on $\bar{y} = \bar{z} = 0$ corresponds an $R_{2n-\alpha-\beta}$ having $x = y = 0$ for $d_{n-\beta}$ and $y = z = 0$ for $g_{n-\alpha-\beta}$. Similarly to an R_n having $x = y = 0$ for d_α and $y = z = 0$ for g_β corresponds a C_{n-} having $\alpha - \beta$ points on $\bar{x} = \bar{y} = 0$ and $n - \alpha - \beta$ points on $\bar{y} = \bar{z} = 0$.

Directrix a Double Line on the Surface.

$$p = 0$$

Fivefold Point.

49. Taking the P_5 and any other point of the nodal curve for fundamental points in depiction I, the R_7 is transformed into a C_5 with a P_3' at one fundamental point and not passing through the other. The scroll of bisecants is an R_{13} whose genus is, in general, 3 but may reduce to 0. To determine how it may break up, take the second fundamental point temporarily on the C_5. The C_5 then depicts into an R_6 with a P_4. From the known properties of the nodal curve of such an R_6,* it is readily seen that the R_{13} can break up only into an R_6 with a d_1 and an R_7 with a d_2.

50. The line joining the fundamental points may cut the C_5 in one point not a fundamental point. The directrix then counts once as a directrix and once as a generator $(d_1 + g_1)$.

* See Wiman, loc. cit. p. 26. Snyder, American Journal of Math., Vol. 27, p. 77.

51. When the R_7 has a g_2, it must pass through the P_5. The fundamental point not on C_5 must then lie on the scroll of bisecants of the C_5. If the R_7 has $2g_2$ then the fundamental point is on the intersection of two generators of the scroll of bisecants (see paragraph 51).

52. When the fundamental point not on C_5 lies on the line joining the points of tangency of two tangents which are coplanar with $\bar{x} = \bar{y} = 0$, then the R_7 has a singularity first noticed by Wiman* which I have called a double torsal generator since it is formed by two coincident torsals having a common torsal plane. The pinch points are, in general, distinct. The line then counts for $2g_2$ as a component of the nodal curve but for only $1g_2$ as a component of the double developable. In the case of the surface we are now considering, however, the pinch points must coincide. Such a $(2g_2)$ counts for $2g_2$ as a component both of the nodal curve and of the double developable.

53. The line on which the fundamental point lies is itself a $(2g_2)$ of the R_{13}. When the fundamental point lies at the pinch point of either torsal of this $(2g_2)$ the multiple line of the R_7 counts, in general, for $3g_2$ as a component of the nodal curve and for $2g_2$ as a component of the double developable. I shall denote it by $(3g_2)$. It should be noticed in the present case, however, that the two pinch points on the $(3g_2)$ coincide and it therefore counts for $3g_2$ as a component of the double developable.

54. Hence we have:

1. $d_2 + C_{14}^2$; $(d_1 + g_1) + C_{14}^2$. The C_{14} has a P'_{10} at the P_5. $p' = 3, 2, 1, 0$ according to the number of discrete P'_2. It meets the directrix four times.

2. $d_2 + 2C_7^2$; $(d_1 + g_1) + 2C_7^2$. Each C_7 has a P'_5 at the P_5 and cuts the other C_7 in $4P'_1$. Each C_7 meets the directrix twice.

3. $d_2 + g_2 + C_{13}^2$. The C_{13} has a P'_9 at the P_5. It meets the g_2 once again. $p' = 3, 2, 1, 0$. It meets the directrix three times.

4. $d_2 + g_2 + C_6^2 + C_7^2$. The C_6 has a P'_4 at the P_5, meets the g_2 once again and the C_7 four times again. The C_7 has a P'_5 at the P_5.

5. $d_2 + 2g_2 + C_{12}^2$. The C_{12} has a P'_8 at the P_5 and meets each g_2 again. $p' = 3, 2, 1, 0$.

6. $d_2 + 2g_2 + C_5^2 + C_7^2$. The C_5 has a P'_3 at P_5 and cuts each g_2 once and the C_7 four times again. The C_7 has a P'_5 at the P_5.

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* Wiman, loc. cit. p. 27.

7. $d_2 + 2g_2 + 2C_6^2$. Each C_6 has a P_4' at the P_5 and meets the other C_6 in four other points. Each cuts $1g_2$ in a P_1'.

8. $d_2 + (2g_2) + C_{12}^2$. The C_{12} has at the P a P_8' with two branches touching the $(2g_2)$. $p' = 2, 1, 0$.

9. $d_2 + (2g_2) + 2C_6^2$. Each C_6 has at the P_5 a P_4' with one branch touching the $(2g_2)$. They meet in $3P_1'$.

10. $d_2 + (3g_2) + C_{11}^2$. The C_{11} has at the P_5 a P_7' with one branch touching the $(3g_2)$. It cuts the latter again with its tangent in the plane through the directrix. $p' = 2, 1, 0$.

11. $d_2 + (3g_2) + C_6^2 + C_5^2$. The C_5 has a P_3' at the P_5, touches the torsal plane of the $(3g_2)$ at another point on the latter and meets the C_6 in $3P_1'$. The C_6 as in (9).

Fourfold Point.

55. Taking the P_4 and a P_3 for fundamental points,* the R_7 is transformed into a C_5 with a P_2' at one fundamental point and a P_1' at the other. The scroll of bisecants is an R_{15} with a d_5. It may be seen from the parametric equations of the R_7 that the R_{15} is of genus 3, 2, 1, 0 and that it can break up only into two unicursal components each having the C_5 for double curve. Since no scroll of degree six or less can have such a C_5^2, the R_{15} can break up only into an R_7 and an R_8. One component has a P_4 and the other a P_3 at the P_2'. If the R_7 had a P_4 there, a line could be drawn through the point meeting the R_7 in eight points. Hence the R_7 has a P_3 there and the R_8 a P_4. The R_7 has $2g_2$ and the R_8 has $4g_2$ one of which passes through the P_2'.

56. The line joining the fundamental points may cut the C_5 in another point giving rise to a (d_1+g_1) or the fundamental point not at the P_2' may be on one, or at the intersection of two trisecants to the C_5 cutting $\bar{x} = \bar{y} = 0$, in which case the R_7 has $1g_2$, or $2g_2$, through the P_4. In particular the two trisecants may be consecutive the R_7 then has a $(2g_2)$ with pinch-points coincident at the P_4.

57. The R_7 may have through the P_4 a $(2g_2)$ the pinch-points of which are not coincident. This happens when the tangent to the C_5, at the fundamental point, cuts the curve at another point and is coplanar with the tangent at the point of intersection and $\bar{x} = \bar{y} = 0$.

*In some particular cases this cannot be done; but the R_7 may still be transformed into such a C_5 by taking the fundamental points elsewhere and following the contact transformation by a point transformation which leaves the linear complex $p_{12} = 0$ invariant.

58. Hence we have:

1. $d_2 + C_{14}^2$; $(d_1 + g_1) + C_{14}^2$. The C_{14} has a P_6' at the P_4 and $6P_3'$. $p' = 3, 2, 1, 0$.

2. $d_2 + 2R_7^2$; $(d_1 + g_1) + 2R_7^2$. Each C_7 has a P_3' at the P_4, P_2' at $3P_3$ and P_1' at the other $3P_3$. They meet in $4P_1'$.

3. $d_2 + g_2 + C_{13}^2$. The C_{13} has a P_5' at the P_4, a P_2' and a P_1' on the g_2 and $5P_3'$. $p' = 3, 2, 1, 0$. The g_2 goes through the P_4 and a P_3.

4. $d_2 + g_2 + C_7^2 + C_6^2$. The C_7 has a P_3' at the P_4, P_2' at $3P_3$ and P_1' at the remaining $2P_3$ not on g_2. It cuts the g_2 in a P_1'. The C_6 has P_2' at the P_4 and at 3 discrete P_3 and P_1' at the other two. They meet in $4P_1'$. The g_2 goes through P_4 and a P_3.

5. $d_2 + g_2 + C_7^2 + C_6^2$. This differs from (4) in that C_6 and C_7 each pass simply through the P_3 on the g_2 and the C_7 cuts it in another P_1'.

6. $d_2 + 2g_2 + C_{12}^2$. The C_{12} has a P_4' at the P_4, P_3 at $4P_3$ and a P_2' and a P_1' on each g_2. $p' = 3, 2, 1, 0$. Each g_2 goes through the P_4 and one P_3.

7. $d_2 + 2g_3 + 2C_6^2$. Each C_6 has a P_2' at the P_4, a P_2 on one g_2 and a P_1' on the other g_2. Each has P_2' at two of the remaining P_3 (which must lie on the same g_1) and passes simply through the other $2P_3$. They meet in $4P_1$. Each g_2 goes through the P_4 and a P_3.

8. $d_2 + (2g_2) + C_{12}^2$. The C_{12} has a P_4' at the P_4 with two branches touching the $(2g_2)$. It touches itself again on the $(2g_2)$ and has P_3 at the 4 discrete P_3. $p' = 2, 1, 0$. The $(2g_2)$ has its pinch-points coincident at the P_4 and passes through $2P_3$ which are consecutive.

9. $d_2 + (2g_2) + 2C_6^2$. Each C_6 has a P_2' at the P_4 with one branch touching the $(2g_2)$. They touch again on the $(2g_2)$. Each has P_2' at two discrete P_3 and P_1' at the other two. The $(2g_2)$ as in (8).

10. $d_2 + (2g_2) + C_{12}^2$. One pinch-point of the $(2g_2)$ is at P_4, the other at a P_3. The C_{12} has a P_4' at the P_4, passes through the other pinch-point of the $(2g_2)$ and touches its torsal plane in two other points on the generator. The C_{12} has a P_3' at each of the 5 discrete P_3. $p' = 2, 1, 0$.

11. $d_2 + (2g_2) + 2C_6^2$. The $(2g_2)$ as in (10). Each C_6 has a P_2' at the P_4, touches the torsal plane of the $(2g_2)$ at another point of it and meets the other C_6 in $3P_1'$. One C_6 has P_2' at $2P_3$ and P_1' at the other four. The other C_6 has P_2' at $3P_3$ and P_1' at the other two discrete P_3.

59. The R_7 may have a g_2 not passing through P_4. It then transforms into a C_5 with $2P_2'$. Such a C_5 may be transformed into an R_6 with a d_1. From

the known properties of the nodal curve of such an R_6 it follows that the R_{14} of bisecants of the C_5 is of genus 3, 2, 1, 0 and may break up into an R_8 and R_6 or into $2R_7$.

60. The fundamental point not at a P'_2 of the C_5 may lie on a trisecant of the C_5 cutting $\bar{x}=\bar{y}=0$, in which case the R_7 has a g_2 through the P_4 in addition to the other. It may also lie at the vertex of a K_2 containing the C_5. According as $\bar{x}=\bar{y}=0$ cuts the K_2 in distinct or consecutive points, the R_7 has $2g_2$ or a $(2g_2)$ with coincident pinch points through the P_4. The R_7 may also have a $(2g_2)$ with distinct pinch-points passing through the P_4 since the configuration on the C_5, as mentioned above, to produce this singularity may exist in this case also.

61. The multiple generator not passing through the P_4 may be a $(2g_2)$ with pinch-points necessarily not coincident.* Both tangents at the corresponding P'_2 of the C_5 then cut $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_{13} of genus 2, 1, 0. It may break up into an R_7 and an R_6. It is seen as before that the original R_7 may have through P_4 a g_2 or a $(2g_2)$ with distinct pinch-points. When the R_7 has $2g_2$, or a $(2g_2)$ with coincident pinch-points, through the P_4, it has no discrete P_3. It may, however, be transformed into one of the C_5 mentioned above by taking a multiple generator through P_4 for $y=z=0$ in transformation II.

62. There exists no R_7 with a P_4 and $2g_2$ neither of which passes through the P_4. For, suppose such an R_7 to exist. Take one of these g_2 for $y=z=0$ in transformation II. A C_5 with a quadrisecant and a P'_2 not on the quadrisecant would then be obtained.

12. $d_2+g_2+C^2_{13}$. The g_2 goes through $3P_3$. The C_{13} has a P'_6 at the P_4, $3P'_2$ and a P'_1 on the g_2 and P'_3 at the other $3P_3$. $p'=3$, 2, 1, 0.

13. $d_2+g_2+C^2_7+C^2_6$. The g_2 goes through $3P_3$. The C_7 has a P'_3 at the P_4, a P'_2 on the g_2 and at each of 2 discrete P_3 and P'_1 at the other $3P_3$. The C_6 has a P'_3 at the P_4. It meets the C_7 twice on the g_2 and cuts the g_2 again. It has a P'_2 at one, and P'_1 at the other two discrete P_3. It meets the C_7 in four other P'_1.

14. $d_2+2g_2+C^2_{12}$. One g_2 goes through the P_4 and a P_3; the other, through $3P_3$. The C_{12} has a P'_5 at the P_4. It has P'_2 at each P_3 on the $2g_2$ and cuts each g_2 in a P'_1. It has P'_3 at the remaining $2P_3$. $p'=3$, 2, 1, 0.

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* They may, however, become consecutive. The corresponding P'_2 of the C_5 is then a cusp.

15. $d_2 + 2g_2 + 2C_6^2$. One g_2 goes through the P_4 and a P_3, the other, through, $3P_3$, One C_6 has a P_2' at the P_4, $2P_1'$ on the g_2 through the P_4, a P_2' and $2P_1'$ on the other g_2, a P_2' at one discrete P_3 and a P_1' at the other. The other C_6 has a P_3' at the P_4. It meets the other C_6 once on the g_2 through the P_4 and twice on the other g_2 and cuts the latter again. It has a P_2' at one discrete P_3 and a P_1' at the other, It meets the other C_6 in four other P_1'.

16. $d_2 + 2g_2 + C_7^2 + C_5^2$. One g_2 goes through the P_4 and a P_3; the other, through $3P_3$. The C_7 has a P_3 at the P_4, and a P_2' on the g_2 through the P_4. It has a P_3' and $2P_1'$ on the other g_2. It has a P_2' at one discrete P_3 and a P_1' at the other. The C_5 has a P_2' at the P_4 and a P_1' on the g_2 through the P_4. It meets the C_7 twice on the other g_2 and cuts this g_2 again. It has a P_2' at one discrete P_3 and a P_1' at the other. It meets the C_7 in four other P_1'.

17. $d_2 + 3g_2 + C_{11}^2$. $2g_2$ go through the P_4 and a P_3, the other goes through $3P_3$. The C_{11} has a P_4' at the P_4, P_2' at the P_3 on the $3g_2$ and a P_3' at the discrete P_3. It has a P_1' on each g_2. $p' = 3, 2, 1, 0$.

18. $d_2 + 3g_2 + C_6^2 + C_5^2$. The $3g_2$ as in (17). The C_6 and C_5 each have a P_2' at P_4, a P_2' on one g_2 through the P_4 and a P_1' on the other. The C_6 meets the third g_2 in a P_2' and $2P_1'$ and has a P_2' at the discrete P_3. The C_5 meets the C_6^2 twice on the g_2 through $3P_3$, cuts this g_2 again, and has a P_1' at the discrete P_3. It meets the C_6 in four other P_1'.

19. $d_2 + (2g_2) + g_2 + C_{11}^2$. The $(2g_2)$ has its pinch-points coincident at the P_4 and goes through $2P_3$. The g_2 goes through $3P_3$. The C_{11} has a P_4' with two branches touching the $(2g_2)$ at P_4. It touches itself again on the $(2g_2)$, cuts the g_2 in $3P_2'$ and a P_1' and has a P_3' at the discrete P_3. $p' = 2, 1, 0$.

20. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The $(2g_2)$ and g_2 as in (19). The C_6 and C_5 each have a P_2' with one branch touching the $(2g_2)$ at P_4. They touch at another point of the $(2g_2)$, intersect twice on the g_2 and in three other P_1'. The C_6 meets the g_2 again in a P_2'. The C_5 meets it in a P_1'. The C_6 has a P_2' and the C_5 a P_1' at the discrete P_3.

21. $d_2 + (2g_2) + g_2 + C_{11}^2$. One pinch-point of the $(2g_2)$ is at P_4, the other at a P_3. The g_2 goes through $3P_3$. The C_{11} has a P_4' at P_4. It has a P_1' at the P_3 on the $(2g_2)$, and touches its torsal plane in two other points of it. It has P_2' at the $3P_3$ on g_2 and P_3' at the discrete $2P_3$. $p' = 2, 1, 0$.

22. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The $(2g_2)$ and g_2 as in (21). The C_6 and C_5 each have a P_2' at P_4, touch the torsal plane of the $(2g_2)$ at another point of it, have a P_2' at one discrete P_3 and a P_1' at the other and meet in three discrete P_1'.

The C_6 has a P_1' at the P_3 on the $(2g_2)$, a P_2' and $2P_1'$ on the g_2. The C_5 has $3P_1'$ on the g_2.

23. $d_2+(2g_2)+C_{12}^2$. The $(2g_2)$ passes through $4P_3$ (one at each pinch-point, two, consecutive, at the intersection with the g_1 in the plane through the $(2g_2)$ and d_1). The C_{12} has a P_6' at P_4 and P_3' at the two discrete P_3. It touches itself on the $(2g_2)$, goes through the pinch-points, and touches the torsal plane of the $(2g_2)$, at two other points of it. $p'=2,\ 1,\ 0$.

24. $d_2+(2g_2)+2C_6^2$. The $(2g_2)$ as in (23). Each C_6 has a P_3' at the P_4, touches the other C_6 on the $(2g_2)$ and meets the $(2g_2)$ twice again. Each has a P_2' at one discrete P_3 and a P_1' at the other. They meet in three other P_1'.

25. $d_2+(2g_2)+g_2+C_{11}^2$. The $(2g_2)$ as in (23). The g_2 goes through the P_4 and a P_3. The C_{11} has a P_5' at the P_4 and a P_2' and a P_1' on the g_2. It has a P_3' at the discrete P_3 and two consecutive P_2' and $4P_1'$ on the $(2g_2)$. $p'=2,\ 1,\ 0$.

26. $d_2+(2g_2)+g_2+C_6^2+C_5^2$. The $(2g_2)$ and g_2 as in (25). The C_6 has a P_3' at the P_4 and a P_2' on the g_2. It has a P_1' at the discrete P_3. The C_5 has a P_2' at the P_4, meets the g_2 again and has a P_2' at the discrete P_3. The C_5 and C_6 touch on the $(2g_2)$ and each meets it in two more points. They meet in three other P_1'.

27. $d_2+(2g_2)+g_2+C_6^2+C_5^2$. This differs from (26) in that the C_6 has a P_1' and not a P_2' on the g_2 and a P_2' and not a P_1' at the discrete P_3 and the C_5 has $2P_1'$ on the g_2 and a P_1' at the discrete P_3.

28. $d_2+(2g_2)+2g_2+C_{10}^2$. The $(2g_2)$ as in (23). Each g_2 goes through the P_4 and a P_3. The C_{10} has a P_4' at the P_4 and a P_2' and a P_1' on each g_2. It has two consecutive P_2' and $4P_1'$ on the $(2g_2)$. $p'=2,\ 1,\ 0$.

29. $d_2+(2g_2)+2g_2+2C_5^2$. The $(2g_2)$ and $2g_2$ as in (28). Each C_5 has a P_2' at the P_4, a P_2' on one g_2 and a P_1' on the other. They touch on the $(2g_2)$ and each cuts the $(2g_2)$ in two more points. They meet in three more P_1'.

30. $d_2+2(2g_2)+C_{10}^2$. One $(2g_2)$ as in (23). The other has its pinch-points coincident at the P_4 and passes through two P_3. The C_{10} has a P_4' with two branches touching the $(2g_2)$ at the P_4. It has two consecutive P_2' on each g_2 and cuts the $(2g_2)$ not passing through the P_4 in four other points. $p'=1,\ 0$.

31. $d_2+2(2g_2)+2C_5^2$. The $2(2g_2)$ as in (30.) Each C_5 has a P_2' at the P_4 with one branch touching the $(2g_2)$. They touch on each $(2g_2)$. Each meets the $(2g_2)$ not passing through the P_4 twice again.

32. $d_2+2\,(2g_2)+C_{10}^2$. One $(2g_2)$ has one pinch-point at the P_4, the other at a P_3. The other $(2g_2)$ passes through $4P_3$. The C_{10} has a P_4' at the P_4, a P_1'

at each of the other three pinch points, and a P_3' at the discrete P_3. It meets each $(2g_2)$ twice again. $p' = 1, 0$.

33. $d_2 + 2\,(2g_2) + 2C_5^2$. The $2\,(2g_2)$ as in (32). Each C_5 has a P_2' at the P_4. One has a P_2 at the discrete P_3. The other P_1' at the discrete P_3 and at the P_3 on the $(2g_2)$ through the P_4. They touch on the other $(2g_2)$ and meet in two discrete P_1'. Each C_5 meets each $(2g_2)$ again.

Ten Threefold Points.

63. From the parametric equations of the surface it is readily seen that the C_{14}^2 of the R_7 is of genus, 3, 2, 1, 0 and that it may break up into two unicursal components each cut by an arbitrary generator twice. At each P_3 one component has a P_2' and the other a P_1' but the second component has through the point a trisecant cutting the directrix. Conversely, on each trisecant of each component which cuts the curve in $3P_1'$ not on the directrix, lies a P_3 of the R_7 at which the other component has a P_2'. Hence, for each component, the sum of the number of these trisecants and of the number of P_2' is ten. The C_{14}^2 can not, therefore, break up into $C_9^2 + C_5^2$ for the maximum of that sum for a C_5 is 8.* Neither can it break up into $C_8^2 + C_6^2$ for the C_6 must have $2P_2'$ and the C_8, $8P_2'$ but such a C_8 must lie on a quadric. The only possible components are, therefore, $2C_7^2$ each having $5P_2'$ and five trisecants of the kind mentioned.

64. It is readily seen that the R_7 may have a g_2 and seven discrete P_3 and that, as before, the double curve may break up. One component has a P_2' and $2P_1'$ on the g_2, the other, $3P_1'$. For each component, therefore, the sum of the P_2' and of the trisecants is eight. If, therefore, the C_{13}^2 broke up into $C_8^2 + C_5^2$, the C_5 could have no P_2' and the C_8 would have to have $8P_2'$ and lie on a quadric. Hence the C_{13}^2 decomposes into $C_7^2 + C_6^2$.

65. The multiple generator may be a $(2g_2)$ passing through $4P_3$. When the C_{12}^2 is composite, the $(2g_2)$ counts for 2 consecutive trisecants on each component. Hence the C_{12}^2 can break up only into $2C_6^2$.

66. From these considerations we obtain:

1. $d_2 + C_{14}^2$; $(d_1 + g_1) + C_{14}^2$. The C_{14} has a P_3' at each P_3. $p' = 3, 2, 1, 0$.

2. $d_2 + 2C_7^2$; $(d_1 + g_1) + 2C_7^2$. Each C_7 has P_2' at $5P_3$ and P_1' at the other $5P_3$. They meet in $4P_1'$. Both are unicursal.

3. $d_2 + g_2 + C_{13}^2$. The C_{13} has $3P_2'$ and a P_1' on the g_2 and P_3' at the seven discrete P_3. $p' = 3, 2, 1, 0$.

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* See paragraph 4.

4. $d_2+g_2+C_7^2+C_6^2$. The C_7 has a P_2' and $2P_1'$ on the g_2, P_2' at four discrete P_3 and P_1' at the other three. The C_6 has $3P_1$ on the g_2, P_2' at two discrete P_3 and P_1' at the other five. It meets the C_7 in four other P_1'. Both are unicursal.

5. $d_2+(2g_2)+C_{12}^2$. The C_{12} touches itself on the $(2g_2)$ and meets the $(2g_2)$ four times again. It has P_3' at the six discrete P_3. $p'=2, 1, 0$.

6. $d_2+(2g_2)+2C_6^2$. Each C_6 touches the other on the $(2g_2)$ and meets the $(2g_2)$ twice again. Each has P_2' at three discrete P_3 and P_1' at the other three. They meet in three other P_1'. Both are unicursal.

67. When the R_7 has $2g_2$, take one of them for $y=z=0$ in transformation II. The R_7 then transforms into a C_5 having a P_2' and having $\bar{y}=\bar{z}=0$ for trisecant. We have seen* that the scroll of bisecants to such a C_5 is an R_{15} which may break up into an R_8 and an R_7. The components of the C_{12}^2 are $C_7^2+C_5^2$ or $2C_6^2$ according as $\bar{y}=\bar{z}=0$ is a g_1 of the R_8 and a g_2 of the R_7 or vice versa.

68. When one of the multiple generators is a $(2g_2)$ both tangents to the C_5 at the P_2' cut $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_{14} of genus 2, 1, 0 which may break up into two R_7. $\bar{y}=\bar{z}=0$ is a g_2 on one component R_7 and a g_1 on the other. When both multiple generators are $(2g_2)$, the C_5 also touches $\bar{y}=0$ at two of its intersections with $\bar{y}=\bar{z}=0$. When the scroll of bisecants is composite, $\bar{y}=\bar{z}=0$ is a multiple generator on each component.

69. When the R_7 has $3g_2$ the C_5 has $2P_2'$. Both tangents at one P_2' may cut $\bar{x}=\bar{y}=0$ in which case the R_7 has $2g_2+(2g_2)$.

7. $d_2+2g_2+C_{12}^2$. The C_{12} has $3P_2'$ and a P_1' on each g_2 and P_3' at the four discrete P_3. $p'=3, 2, 1, 0$.

8. $d_2+2g_2+2C_6^2$. Each C_6 has a P_2' and $2P_1'$ on one g_2 and $3P_1'$ on the other. Each has P_2' at two discrete P_3 and P_1' at the other two. They meet in four other P_1'. Both are unicursal.

9. $d_2+2g_2+C_7^2+C_5^2$. The C_7 has a P_2' and $2P_1'$ on each g_2, a P_2' at three discrete P_3 and a P_1' at the other. The C_5 has $3P_1'$ on each g_2, a P_2' at one discrete P_3 and P_1' at the other three. The C_5 and C_7 meet in four other P_1'. Both are unicursal.

10. $d_2+(2g_2)+g_2+C_{11}^2$. The C_{11} has a $3P_2'$ and a P_1' on the g_2. It touches itself on the $(2g_2)$ and meets the $(2g_2)$ four times again. It has P_3' at the three discrete P_3. $p'=2, 1, 0$.

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*See paragraph 55.

11. $d_2 + (2g_2) + g_2 + C_6^2 + C_5^2$. The C_6 has a P_2' and $2P_1'$ on the g_2; $3P_1'$ on the $(2g_2)$, P_2' at two discrete P_3 and a P_1' at the other. The C_5 cuts each multiple generator thrice, has a P_2' at one discrete P_3 and a P_1' at the other two. It meets the C_6 in three other P_1'. Both are unicursal.

12. $d_2 + 2\,(2g_2) + C_{10}^2$. The C_{10} has two consecutive P_2' and $4P_1'$ on each $(2g_2)$, and P_3' at the two discrete P_3. $p' = 1$, or 0.

13. $d_2 + 2(2g_2) + 2C_5^2$. Each C_5 touches the other on each $(2g_2)$ and meets each $(2g_2)$ twice again. Each has a P_2' at one discrete P_3, and a P_1' at the other. They meet in two more P_1'. Both are unicursal.

14. $d_2 + 3g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on each g_2 and a P_3' at the discrete P_3. $p' = 3, 2, 1, 0$.

15. $d_2 + 3g_2 + C_6^2 + C_5^2$. The C_6 has a P_2' and $2P_1'$ on each of $2g_2$ and $3P_1'$ on the other. It has a P_2' at the discrete P_3. The C_5 has a P_2' and $2P_1'$ on $1g_2$ and $3P_1'$ on each of the other two. It has a P_1' at the discrete P_3 and cuts the C_6 in four more P_1'. Both curves are unicursal.

16. $d_2 + (2g_2) + 2g_2 + C_{10}^2$. The C_{10} has $3P_2'$ and a P_1' on each g_2 and $2P_2'$ and $4P_1'$ on the $(2g_2)$. $p' = 2, 1, 0$.

17. $d_2 + (2g_2) + 2g_2 + 2C_5^2$. Each C_5 has a P_2' and $2P_1'$ on one g_2 and $3P_1'$ on the other g_2. They touch on the $(2g_2)$ and each meets the $(2g_2)$ twice again. They meet in three other P_1'. Each is unicursal.

$$p = 1.$$

Fourfold point.

70. The R_7 is transformed by I into a C_5 of genus 1, having a P_2'. The scroll of bisecants is an R_{14}. The fundamental point not at P_2' may be on a trisecant of the C_5 cutting $\bar{x} = \bar{y} = 0$. It may also be at the vertex of a K_2 which contains the C_5 and cuts $\bar{x} = \bar{y} = 0$ in two distinct or coincident points.

71. The R_7 cannot have a g_2 not passing through the P_4 for then it would transform by II into a C_5 of genus 1 having a quadrisecant.

The double curve cannot break up. For, the cone having its vertex at P_4 and containing one component would have to be unicursal. The component itself would, therefore, be unicursal. This is impossible.

1. $d_2 + C_{13}^2$. The C_{13} has a P_6' at the P_4 and P_3' at the $3P_3$. p' is, at most, 6.

2. $d_2 + g_2 + C_{12}^2$. The g_2 goes through the P_4. The C_{12} has a P_5' at the

P_4 and meets the g_2 again in a P_2' and a P_1'. It has P_3' at the two discrete P_3. Its maximum genus is 6.

3. $d_2 + (2g_2) + C_{11}^2$. The $(2g_2)$ has its pinch-points coincident at the P_4. The C_{11} has a P_4' with two branches touching the $(2g_2)$ at the P_4, touches itself again on the $(2g_2)$ and has a P_3' at the discrete P_3. Its maximum genus is 5.

4. $d_2 + 2g_2 + C_{11}^2$. Each g_2 goes through the P_4 and a P_3. The C_{11} has a P_4' at the P_4, a P_2' and a P_1' on each g_2 and a P_3' at the discrete P_3. Its maximum genus is 6.

Seven threefold points.

72. The R_7 is transformed by I into a C_6 of genus 1 with $2P_2'$. When the C_6 has a third P_2' the R_7 has a g_2 or a $(2g_2)$. When the R_7 has $2g_2$ it is transformed by II into such a C_5 as was found for the transform in the case of a P_4. One, but not both, multiple generators may become double torsal. The double curve is not decomposable.

1. $d_2 + C_{13}^2$. The C_{13} has P_3' at the $7P_3$. Its maximum genus is 6.

2. $d_2 + g_2 + C_{12}^2$. The C_{12} has $3P_2'$ and a P_1' on the g_2 and P_3' at the 4 discrete P_3. Its maximum genus is 6.

3. $d_2 + (2g_2) + C_{11}^2$. The C_{11} has 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$ and P_3' at the 3 discrete P_3. Its maximum genus is 5.

4. $d_2 + 2g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on each g_2 and a P_3' at the discrete P_3. Its maximum genus is 6.

5. $d_2 + (2g_2) + g_2 + C_{10}^2$. The C_{10} has $3P_2'$ and a P_1' on the g_2 and 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$. Its maximum genus is 5.

$$p = 2.$$

73. The R_7 has $4P_3$. The double curve is not decomposable.

1. $d_2 + C_{12}^2$. The C_{12} has P_3' at the $4P_3$. Its maximum genus is 9.

2. $d_2 + g_2 + C_{11}^2$. The C_{11} has $3P_2'$ and a P_1' on the g_2 and a P_3' at the discrete P_3. Its maximum genus is 9.

3. $d_2 + (2g_2) + C_{10}^2$. The C_{10} has 2 consecutive P_2' and $4P_1'$ on the $(2g_2)$. Its maximum genus is 8.

$$p = 3.$$

74. The R_7 has one P_3. The double curve is not decomposable.

1. $d_2 + C_{11}^2$. The C_{11} has a P_3' at P_3. Its maximum genus is 12.

Directrix a threefold line on the surface.

$p = 0.$

Fourfold point.

75. The R_7 is transformed by I into a C_4 of the second kind passing through one fundamental point. Transform the C_4 by II into an R_5. From the known properties of the nodal curve of such an R_5,* we find that the R_9 of bisecants of the C_4 may break up into an R_6 and an R_3 or into $3R_3$. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1, 2 points besides the fundamental point on the curve.

76. When the fundamental point not on the C_4 is on the scroll of bisecants the R_7 has a g_2 through the P_4. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1 points besides the fundamental point on the C_4. It cannot meet it in two more points for then it would meet 10 generators of the scroll of bisecants.

77. Similarly, this fundamental point may be at the intersection of two generators of the scroll of bisecants, giving rise to $2g_2$ through the P_4. When the C_4 has two tangents coplanar with the directrix the fundamental point may lie on a $(2g_2)$ of the scroll of bisecants (in case the scroll of bisecants is composite these will be coincident simple torsals of two components) giving rise to a $(2g_2)$ of the R_7 with pinch-points coincident at the P_4. In particular, the fundamental point may lie at a pinch-point of the $(2g_2)$ of the scroll of bisecants. The R_7 then has a $(3g_2)$ through the P_4.

1. $d_3 + C_{12}^2$; $(d_2 + g_1) + C_{12}^2$; $(d_1 + 2g_1) + C_{12}^2$. The C_{12} has a P_6' at the P_4 and P_3' at the $3P_3$. $p' = 1, 0$.

2. $d_3 + C_8^2 + C_4^2$; $(d_2 + g_1) + C_8^2 + C_4^2$; $(d_1 + 2g_1) + C_8^2 + C_4^2$. The C_8 has a P_4' at the P_4 and P_2' at the $3P_3$. The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. It meets the C_8 in $2P_1'$.

3. $d_3 + 2C_4^2$; $(d_2 + g_1) + 3C_4^2$; $(d_1 + 2g_1) + 3C_4^2$. Each C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. Each two C_4 meet in another P_1'.

4. $d_3 + g_2 + C_{11}^2$; $(d_2 + g_1) + g_2 + C_{11}^2$. The g_2 goes through the P_4. The C_{11} has a P_5' at the P_4 and P_3' at the $3P_3$. It meets the g_2 again in a P_1'. $p' = 1, 0$.

5. $d_3 + g_2 + C_7^2 + C_4^2$; $(d_2 + g_1) + g_2 + C_7^2 + C_4^2$. The g_2 goes through the P_4. The C_7 has a P_3' at the P_4, P_2' at the $3P_3$, a P_1' on the g_2 and $2P_1'$ on the C_4. The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$.

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* See Snyder, On the forms of quintic scrolls, Bulletin of Am. Math. Society, 2d Series, Vol. VIII, p. 293.

6. $d_3+g_2+C_8^2+C_3^2$; $(d_2+g_1)+C_8^2+C_3^2$. The g_2 goes through the P_4. The C_8 has a P_4' at the P_4 and P_2' at the $3P_3$. The C_3 is gauche, has P_1' at the P_4 and at the $3P_3$. It meets the g_2 again in a P_1' and the C_8 in $2P_1'$.

7. $d_3+g_2+2C_4^2+C_3^2$; $(d_2+g_1)+g_2+2C_4^2+C_3^2$. The g_2 goes through the P_4. Each C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. Each meets the other C_4 and the C_3 in another P_1'. The C_3 is gauche, has P_1' at the P_4 and at the $3P_3$ and meets the g_2 again in a P_1'.

8. $d_3+2g_2+C_{10}^2$; $(d_2+g_1)+2g_2+C_{10}^2$. The $2g_2$ pass through the P_4. The C_{10} has a P_4' at the P_4 and P_3' at the $3P_3$. $p'=1, 0$.

9. $d_3+2g_2+C_6^2+C_4^2$; $(d_2+g_1)+2g_2+C_6^2+C_4^2$. The $2g_2$ pass through the P_4. The C_6 has P_2' at the P_4 and at the $3P_3$ and a P_1' on each g_2. The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. The C_6 and C_4 meet in $2P_1'$.

10. $d_3+2g_2+C_8^2+C_2^2$. The $2g_2$ pass through the P_4. The C_8 has a P_4' at the P_4 and P_2' at the $3P_3$. The C_2 has P_1' at the $3P_3$, a P_1' on each g_2 and meets the C_8 in $2P_1'$. It does not pass through the P_4.

11. $d_3+2g_2+2C_4^2+C_2^2$. The $2g_2$ pass through the P_4. Each C_4 has a P_2' at the P_4 and P_3' at the $3P_3$. Each meets the other C_4 and the C_2 in a P_1'. The C_2 has a P_1' on each g_2 and P_1' at the $3P_3$.

12. $d_3+(2g_2)+C_{10}^2$; $(d_2+g_1)+(2g_2)+C_{10}^2$. The $(2g_2)$ goes through the P_4. The C_{10} has P_3' at the $3P_3$. At the P_4 it has a P_4' with two branches touching the $(2g_2)$. $p'=0$.

13. $d_3+(2g_2)+C_7^2+C_3^2$; $(d_2+g_1)+(2g_2)+C_7^2+C_3^2$. The $(2g_2)$ goes through the P_4. The C_7 has, at the P_4, a P_3' with one branch touching the $(2g_2)$. It has a P_2' at each P_3. The C_3 is gauche, touches the $(2g_2)$ at the P_4, has P_1' at each P_3 and meets the C_7 again in a P_1'.

14. $d_3+(2g_2)+C_4^2+2C_3^2$; $(d_2+g_1)+(2g_2)+C_4^2+2C_3^2$. The $(2g_2)$ goes through the P_4. The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. Each C_3 is gauche, touches the $(2g_2)$ at the P_4, has P_1' at the $3P_3$ and meets the C_4 again in a P_1'.

15. $d_3+(3g_2)+C_9^2$; $(d_2+g_1)+(3g_2)+C_9^2$. The $(3g_2)$ goes through the P_4. The C_9 has P_3' at the P_4 and at the $3P_3$. It touches the $(3g_2)$ at the P_4 and touches its torsal plane at another point of it. $p'=0$.

16. $d_3+(3g_2)+C_6^2+C_3^2$; $(d_2+g_1)+(3g_2)+C_6^2+C_3^2$. The $(3g_2)$ goes through the P_4. The C_6 has P_2' at the P_4 and at the $3P_3$. It touches the torsal plane of the $(3g_2)$ at another point of it. The C_3 is gauche, touches the $(3g_2)$ at the P_4 and has P_1' at the $3P_3$. It meets the C_6 in a P_1'.

17. $d_3 + (3g_2) + C_7^2 + C_2^2$. The $(3g_2)$ goes through the P_4. The C_7 has, at the P_4, a P'_3 with one branch touching the $(3g_2)$. It has a P'_2 at each P_3 and meets the C_2 in a P'_1. The C_2 touches the torsal plane of the $(3g_2)$ at a point of it and has P'_1 at the $3P_3$.

18. $d_3 + (3g_2) + C_4^2 + C_3^2 + C_2^2$. The $(3g_2)$ goes through the P_4. The C_4 has a P'_2 at the P_4, P'_1 at the $3P_3$ and meets the C_3 and C_2 each again in a P'_1. The C_3 is gauche, touches the $(3g_2)$ at the P_4 and has P'_1 at the $3P_3$. The C_2 touches the torsal plane of the $(3g_2)$ at a point of it and has P'_1 at the $3P_3$.

78. When the scroll of bisecants of the C_4 into which the R_7 is transformed has a component R_3, the d_2 of this R_3 may be taken for $\bar{z} = \bar{w} = 0$. The fundamental point on the C_4 then lies on a trisecant of the C_4 cutting $\bar{x} = \bar{y} = 0$. It therefore gives rise on the R_7 to a g_2 not passing through the P_4. The fundamental point not on the C_4 is either at the intersection of $2g_1$ or at a pinch-point of a $(2g_2)$ of the scroll of bisecants and gives rise correspondingly to two intersecting g_2 or to a $(3g_2)$. The C_4 has $2P'_1$ on $\bar{z} = \bar{w} = 0$. One of them is a fundamental point; the other transforms into a g_1 coincident with the directrix. The two remaining generators which pass through a point of the directrix also lie in a plane through the directrix since they are the transforms of points lying on a complex line cutting $\bar{z} = \bar{w} = 0$. The directrix is, therefore, a contact directrix* with which a generator coincides, $(\delta_{2,1} + g_1)$. An R_7 is the simplest surface that can have this singularity apart from the restricted case of cubics and quintics contained in a special linear congruence.

79. In general when C lies on an R having $\bar{x} = \bar{y} = 0$ and $\bar{z} = \bar{w} = 0$ for directrices and when the C has j points other than fundamental points on $\bar{z} = \bar{w} = 0$ and has k points on each of the l generators of the R which lie in an arbitrary plane through $\bar{z} = \bar{w} = 0$, then the C transforms into an R which has $x = y = 0$ for $(\delta_{k,1} + jg_1)$.

19. $(\delta_{2,1} + g_1) + 3g_2 + C_8^2$. $2g_2$ pass through the P_4; the other, through $2P_3$. The C_8 has a P'_4 at the P_4 and a P'_2 at each P_3 of which two are on a g_2 and the other necessarily on the directrix. It meets the directrix in two more P'_1.

20. $(\delta_{2,1} + g_1) + 3g_2 + 2C_4^2$. $2g_2$ pass through the P_4; the other through $2P_3$. Each C_4 has a P'_2 at the P_4 a P'_1 at each P_3 (of which one is on the directrix) and cuts the directrix once again.

21. $(\delta_{2,1} + g_1) + (3g_2) + g_2 + C_7^2$. The $(3g_2)$ passes through the P_4; the g_2 through $2P_3$. The C_7 has at the P_4 a P'_3 with one branch touching the $(3g_2)$. It

* Wiman, loc. cit. p. 43.

has P_2' at the $3P_3$ (of which one is on the directrix) and meets the directrix in two other points.

22. $(\delta_{2,1} + g_1) + (3g_2) + g_2 + C_4^2 + C_3^2$. The $(3g_2)$ and g_2 as in (21). The C_4 has a P_2' at the P_4, P_1' at the $3P_3$ (of which one is on the directrix) and cuts the directrix again. The C_3 is gauche, touches the $(3g_2)$ at the P_4, has P_1' at the $3P_3$ and meets the C_4 in another P_1'.

80. The remaining cases in which the R_7 has a g_2 not passing through the P_4 may be more conveniently transformed into C_4 having a P_2'. Take temporarily for fundamental points the P_2' and a P_1' and transform the C_4 into an R_5. From the know properties of the nodal curve of such an R_5,* we find that the R_8 of bisecants to the C_4 may break up into an R_5 and an R_3, or into an R_6 and a K_2, or into $2R_3$ and a K_2.

81. In addition to the fundamental point on the C_4, $\bar{z} = \bar{w} = 0$ may cut the C_4 in a P_1' or at the P_2'. In the latter case, the R_7 has two generators with coincident torsals coinciding with the directrix. As the directrix now counts for four instead of for three double lines as a component of the nodal curve, I have denoted this singularity by $(d_1 + g_2)$. It should be noticed, however, that the singularity is analogous to a $(2g_2)$ and it differs from both the g_2 and $(2g_2)$ in that it may occur, as in this case, on a line which counts only once as a directrix.

22. The fundamental point not on the C_4 may lie on one or on two g_1 of the scroll of bisecants, giving rise to one g_2 or two g_2 through the P_4. It may also lie on a $(2g_2)$ or, in particular, at a pinch-point of a $(2g_2)$ of the scroll of bisecants, giving rise to a $(2g_2)$ or a $(3g_2)$ through the P_4 on the R_7.

23. $d_3 + g_2 + C_{11}^2$; $(d_2 + g_1) + g_2 + C_{11}^2$; $(d_1 + g_2) + C_{11}^2$. The C_{11} has a P_6' at the P_4, P_2' at the $2P_3$ on the g_2 and a P_3' at the discrete P_3. $p' = 1, 0$.

24. $d_3 + g_2 + C_8^2 + C_3^2$; $(d_2 + g_1) + g_2 + C_8^2 + C_3^2$; $(d_1 + g_2) + C_8^2 + C_3^2$. The C_8 has a P_4' at the P_4, P_2' at the $3P_3$ and meets the C_3 in $2P_1'$. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the g_2.

25. $d_3 + g_2 + C_7^2 + C_4^2$; $(d_2 + g_1) + g_2 + C_7^2 + C_4^2$; $(d_1 + g_2) + C_7^2 + C_4^2$. The C_7 has a P_4' at the P_4, a P_2' at the discrete P_3 and $3P_1'$ on the g_2. The C_4 has a P_2' at the P_4, a P_1' at the discrete P_3, meets the C_7 in the $2P_3$ on the g_2 and in two other P_1'.

26. $d_3 + g_2 + 2C_4^2 + C_3^2$; $(d_2 + g_1) + g_2 + 2C_4^2 + 2C_3^2$; $(d_1 + g_2) + 2C_4^2 + C_3^2$. Each C_4 has a P_2' at the P_4, P_1' at the $3P_3$ and meets the other C_4 and the C_3

* See Snyder, On the forms of quintic scrolls, Bulletin of the Amer. Math. Soc., 2d Series, Vol. 8, 293.

each again in a P_1'. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the g_2.

27. $d_3 + 2g_2 + C_{10}^2$; $(d_2 + g_1) + 2g_2 + C_{10}^2$. One g_2 goes through the P_4; the other, through $2P_3$. The C_{10} has a P_5' at the P_4, a P_3' at the discrete P_3 and a $2P_2'$ and a P_1' on the g_2. $p' = 1, 0$.

28. $d_3 + 2g_2 + C_8^2 + C_2^2$. The $2g_2$ as in (27). The C_8 has a P_4' at the P_4 and P_2' at the $3P_3$. It meets the C_2 in $2P_1'$. The C_2 has P_1' at the P_4 and at the discrete P_3. It meets each g_2 once.

29. $d_3 + 2g_2 + C_7^2 + C_3^2$; $(d_2 + g_1) + 2g_2 + C_7^2 + C_3^2$. The $2g_2$ as in (27). The C_7 has a P_3' at the P_4 and a P_2' at the $3P_3$. It meets the g_2 through the P_4 in a P_1' and the C_3 in $2P_1'$. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the g_2 not passing through the P_4. (This C_3 transforms into a K_2. The one in (30) transforms into an R_3.)

30. $d_3 + 2g_2 + C_7^2 + C_3^2$; $(d_2 + g_1) + 2g_2 + C_7^2 + C_3^2$. The $2g_2$ as in (27). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_3 and $3P_1'$ on the g_2 not passing through the P_4. The C_3 has a P_1' at each multiple point, meets the C_7 once and the g_2 through the P_4 twice, again.

31. $d_3 + 2g_2 + C_6^2 + C_4^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_4^2$. The $2g_2$ as in (27). The C_6 has a P_3' at the P_4 and a P_2' at the discrete P_3. It meets the g_2 through the P_4 in a P_1' and the other g_2 in $3P_1'$. The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. It meets the C_6 in $2P_1'$.

32. $d_3 + 2g_2 + 2C_4^2 + C_2^2$. The $2g_2$ as in (27). Each C_4 has a P_2' at the P_4, P_1' at the $3P_3$ and meets the other C_4 and the C_2 in another P_1'. The C_2 has a P_1' at the P_4 and at the discrete P_3.

33. $d_3 + 2g_2 + C_4^2 + 2C_3^2$; $(d_2 + g_1) + 2g_2 + C_4^2 + 2C_3^2$. The $2g_2$ as in (27). The C_4 has a P_2' at the P_4 and a P_1' at each P_3. One C_3 has P_1' at the P_4 and at the $3P_3$ and meets the g_2 through the P_4 again. The other C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and has a P_1' on the g_2 passing through $2P_3$. Each C_3 meets the C_4 and the other C_3 again in a P_1'.

34. $d_3 + 3g_2 + C_9^2$; $(d_2 + g_1) + 3g_2 + C_9^2$. $2g_2$ pass through the P_4; the other through $2P_3$. The C_9 has a P_4' at the P_4, a P_3' at the discrete P_3 and P_2' at the other $2P_3$. It meets each g_2 in a P_1'. $p' = 1, 0$.

35. $d_3 + 3g_2 + C_7^2 + C_2^2$. The $3g_2$ as in (34). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_3 and $3P_1'$ on the g_2 not passing through the P_4. The C_2 has P_1' at the $3P_3$, meets the C_7 in $2P_1'$ and has a P_1' on each g_2 through the P_4.

36. $d_3 + 3g_2 + C_6^2 + C_3^2$. The $3g_2$ as in (34). The C_6 has P_2' at the P_4 and at the $3P_3$, and meets each g_2 through the P_4 again. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and meets the C_6 in two other P_1'. It has a P_1' on the g_2 passing through $2P_3$.

37. $d_3 + 3g_2 + C_5^2 + C_4^2$. The $3g_2$ as in (34). The C_5 has P_2' at the P_4 and at the discrete P_3. It meets each g_2 through the P_4 in a P_1' and the other g_2 in $3P_1'$. The C_4 has a P_2' at the P_4, P_1' at the $3P_3$ and two other P_1' on the C_5.

38. $d_3 + 3g_2 + C_4^2 + C_3^2 + C_2^2$. The $3g_2$ as in (34). The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the g_2 through $2P_3$. The C_2 has P_1' at the $3P_3$ and a P_1' on each g_2 through the P_4. Each pair of the curves C_4, C_3 and C_2 have another P_1' in common.

39. $d_3 + (2g_2) + g_2 + C_9^2$. The $(2g_2)$ has its pinch-points coincident at the P_4. The g_2 goes through $2P_3$. The C_9 has at the P_4 a P_4' with two branches touching the $(2g_2)$. It has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on the g_2. $p' = 0$.

40. $d_3 + (2g_2) + g_2 + C_7^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (39). The C_7 has at the P_4 a P_3' with one branch touching the $(2g_2)$. It has P_2' at the $3P_3$. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the discrete P_3 and meets the C_7 again. It has a P_1' on the g_2.

41. $d_3 + (2g_2) + g_2 + C_6^2 + C_3^2$. The $(2g_2)$ and the g_2 as in (39). The C_6 has, at the P_4, a P_3' with one branch touching the $(2g_2)$. It has a P_2' at the discrete P_3 and $3P_1'$ on the g_2. The C_3 touches the $(2g_2)$ at the P_4, has P_1' at the $3P_3$ and meets the C_6 again.

42. $d_3 + (2g_2) + g_2 + C_4^2 + C_3^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (39). The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. The C_3 touches the $(2g_2)$ at the P_4 and has P_1' at the $3P_3$. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the discrete P_3 and meets the g_2. The C_4 meets the C_3 and the C_2 each in another P_1'.

43. $d_3 + (2g_2) + g_2 + 3C_3^2$. The $(2g_2)$ and the g_2 as in (39). One C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and meets the g_2. Each of the other $2C_3$ touches the $(2g_2)$ at the P_4 and has P_1' at the $3P_3$. The first C_3 meets each of the others again.

44. $d_3 + (3g_2) + g_2 + C_8^2$. The pinch-points on the $(3g_2)$ are coincident at the P_4. The g_2 goes through $2P_3$. The C_8 has, at the P_4, a P_3' with one branch

touching the $(3g_2)$. It touches the torsal plane of the $(3g_2)$ at another point of it. The C_8 has a P_2' at the discrete P_3 and $2P_2'$ and a P_1' on the g_2. $p'=0$.

45. $d_3+(3g_2)+g_2+C_6^2+C_2^2$. The $(3g_2)$ and the g_2 as in (44). The C_6 has P_2' at the P_4 and at the $3P_3'$ and touches the torsal plane of the $(3g_2)$ at another point of it. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the discrete P_3 and meets the g_2. It meets the C_6 again in a P_1'.

46. $d_3+(3g_2)+g_2+C_6^2+C_2^2$. The $(3g_2)$ and the g_2 as in (44). The C_6 has at the P_4 a P_3' with one branch touching $(3g_2)$. It has a P_2' at the discrete P_3 and a $3P_1'$ on the g_2. The C_2 has a P_1' at the discrete P_3, meets the C_6 in $2P_1'$ on the g_2 and in another P_1' and touches the torsal plane of the $(3g_2)$ at a point of the $(3g_2)$.

47. $d_3+(3g_2)+g_2+C_5^2+C_3^2$. The $(3g_2)$ and the g_2 as in (44). The C_5 has a P_2' at the P_4 and touches the torsal plane of the $(3g_2)$ at another point of the latter. It has a P_2' at the P_3 and $3P_1'$ on the g_2. The C_3 touches the $(3g_2)$ at P_4, has a P_1' at the P_3 and meets the C_5 in $2P_1'$ on the g_2 and in another P_1'.

48. $d_3+(3g_2)+g_2+C_4^2+2C_2^2$. The $(3g_2)$ and the g_2 as in (44). The C_4 has a P_2' at the P_4 and P_1' at the $3P_3$. One C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the discrete P_3 and meets the g_2. The other has P_1' at the $3P_3$ and touches the torsal plane of the $(3g_2)$ at a point of the latter. The C_4 meets each in another P_1'.

49. $d_3+(3g_2)+g_2+2C_3^2+C_2^2$. The $(3g_2)$ and the g_2 as in (44). One C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and meets the g_2. The other C_3 touches the $(3g_2)$ at the P_4 and has P_1' at the $3P_3$. The C_2 has P_1' at the $3P_3$ and touches the torsal plane of the $(3g_2)$ at a point of the latter. The first C_3 meets the second C_3 and the C_2 each in another P_1'.

83. When the R_7 has a $(2g_2)$ not passing through the P_4, it may be transformed by I into a C_4 with a P_2' at which the tangent cuts $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_7 of genus zero, which may break up into an R_5 and a K_2 or into an R_3 and $2K_2$.

84. In addition to the fundamental point on the C_4 the line $\bar{z}=\bar{w}=0$ may cut the C_4 in a P_1' or at the P_2'. In the latter case an arbitrary plane section of the corresponding R_7 has, at the trace of $x=y=0$, a P_3' at which two branches have contact of the second order. $x=y=0$ counts for five as a component of the nodal curve. I shall denote the singularity by $[d_1+(2g_2)]$.

85. As before, it is seen that the R_7 may have through the P_4 a g_2, $2g_2$, a $(2g_2)$, or a $(3g_2)$.

50. $d_3 + (2g_2) + C_{10}^2$; $(d_2 + g_1) + (2g_2) + C_{10}^2$; $[d_1 + (2g_2)] + C_{10}^2$. The $(2g_2)$ goes through $2P_3$. The C_{10} has a P_6' at the P_4, a P_3' at the discrete P_3 and four points on the $(2g_2)$. $p' = 0$.

51. $d_3 + (2g_2) + C_7^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + C_7^2 + C_3^2$; $[d_1 + (2g_2)] + C_7^2 + C_3^2$. The $(2g_2)$ as in (50). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_3 and $3P_1'$ on the $(2g_2)$. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. It meets the C_7 again in a P_1'.

52. $d_3 + (2g_2) + C_4^2 + 2C_3^2$; $(d_2 + g_1) + (2g_2) + C_4^2 + 2C_3^2$; $[d_1 + (2g_2)] + C_4^2 + 2C_3^2$. The $(2g_2)$ as in (50). The C_4 has a P_2' at the P_4 and a P_1' at the $3P_3$. Each C_3 has a P_2' at the P_4, a P_1' at the discrete P_3, another P_1' on the C_4 and a P_1' on the $(2g_2)$.

53. $d_3 + (2g_2) + g_2 + C_9^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_9^2$. The $(2g_2)$ goes through $2P_3$; the g_2, through the P_4. The C_9 has a P_5' at the P_4 and a P_3' at the discrete P_3. It meets the g_2 again and has $4P_1'$ on the $(2g_2)$. $p' = 0$.

54. $d_3 + (2g_2) + g_2 + C_7^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (53). The C_7 has a P_4' at the P_4, a P_2' at the discrete P_3 and $3P_1'$ on the $(2g_2)$. The C_2 has P_1' at the P_4 and at the discrete P_3, meets C_7 and the g_2 each again and has a P_1' on the $(2g_2)$.

55. $d_3 + (2g_2) + g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_6^2 + C_3^2$. The $(2g_2)$ and the g_2 as in (53). The C_6 has a P_3' at the P_4, a P_2' at the discrete P_3 and P_1' at the $2P_3$ on the $(2g_2)$. It meets the $(2g_2)$, the g_2 and the C_3 each again. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$.

56. $d_3 + (2g_2) + g_2 + C_4^2 + C_3^2 + C_2^2$. The $(2g_2)$ and the g_2 as in (53). The C_4 has a P_2' at the P_4, P_1' at the $3P_3$ and meets the C_3 and C_2 each again. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. The C_2 has a P_1' at the P_4 and at the discrete P_3, meets the g_2 again and meets the $(2g_2)$.

57. $d_3 + (2g_2) + g_2 + 3C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + 3C_3^2$. The $(2g_2)$ and the g_2 as in (53). Two C_3 have each a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. The other C_3 is gauche, has a P_1' at the P_4, P_1' at the $3P_3$ and meets the g_2 and the other $2C_3$ each again.

58. $d_3 + (2g_2) + 2g_2 + C_8^2$. The $(2g_2)$ passes through $2P_3$; each g_2, through the P_4. The C_8 has a P_4' at the P_4 and a P_3' at the discrete P_3. It meets each g_2 in a P_1' and the $(2g_2)$ in $4P_1'$.

59. $d_3 + (2g_2) + 2g_2 + C_5^2 + C_3^2$. The $(2g_2)$ and the $2g_2$ as in (58). The C_5 has P_2' at the P_4 and at the discrete P_3. It has a P_1' on each g_2 and $3P_1'$ on the

$(2g_2)$. The C_3 has a P_2' at the P_4 and a P_1' at the discrete P_3. It meets the C_5 again and meets the $(2g_2)$ once.

60. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. The $(2g_2)$ and the $2g_2$ as in (58). Each C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. The C_2 has P_1' at the $3P_3$, meets each C_3 again and has a P_1' on each g_2.

61. $d_3 + 2\,(2g_2) + C_6^2 + C_2^2$. One $(2g_2)$ has its pinch-points coincident at the P_4; the other has two pinch-points which are P_3 on the R_7. The C_6 has, at the P_4, a P_3' with one branch touching the $(2g_2)$. It has $3P_1'$ on the other $(2g_2)$ and a P_2' at the discrete P_3. The C_2 touches one $(2g_2)$ at the P_4, meets the other $(2g_2)$ in a P_1' and has a P_1' at the discrete P_3. It does not meet C_6 again.

62. $d_3 + 2\,(2g_2) + 2C_3^2 + C_2^2$. The $2(2g_2)$ as in (61). One C_3 touches one $(2g_2)$ at the P_4 and has P_1' at the $3P_3$. The other C_3 has a P_2' at the P_4, a P_1' at the discrete P_3, meets the other C_3 again and meets the $(2g_2)$ through $2P_3$ once. The C_2 touches one $(2g_2)$ at the P_4, meets the other $(2g_2)$ once and has a P_1' at the discrete P_3.

63. $d_3 + (3g_2) + (2g_2) + C_5^2 + C_2^2$. The $(3g_2)$ passes through the P_4; the $(2g_2)$ through $2P_3$. The C_5 has P_2' at the P_4 and at the discrete P_3. It meets the $(3g_2)$ again and has $3P_1'$ on the $(2g_2)$. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the discrete P_3 and meets the $(2g_2)$ once.

64. $d_3 + (3g_2) + (2g_2) + C_3^2 + 2C_2^2$. The C_3 has a P_2' at the P_4, a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. One C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the discrete P_3 and a P_1' on the $(2g_2)$. The other C_2 has P_1' at the $3P_3$, meets the C_3 again and meets the $(3g_2)$ once.

86. When the R_7 has g_3, we may take the g_3 for $y = z = 0$ in transformation II and transform the R_7 into a C_4 of the second kind having a P_1' on $\bar{y} = \bar{z} = 0$. The R_7 has only $2P_3$ since only two trisecants of the C_4 cut $\bar{x} = \bar{y} = 0$. The scroll of bisecants is, as we have seen,* an R_9 which may break up into an R_6 and an R_3 or into $3R_3$. The three generators of the scroll of bisecants through the P_1' of the C_4 on $\bar{y} = \bar{z} = 0$ transform into the triple point of the double curve at the P_4; the other three generators in $\bar{y} = 0$ transform into three other points on the g_3.

87. When the R_7 has a g_2 in addition to the g_3, the C_4 has a P_2'. The scroll of bisecants in an R_8 which may break up into an R_6 and a K_2, or into an R_5 and

* See paragraph 75.

an R_3 or into $2R_3$ and a K_2. When the P_2' lies in $\bar{y}=0$ the g_2 becomes consecutive to the g_3.

88. When the tangents to the P_2' cut $\bar{x}=\bar{y}=0$, the R_7 has a $(2g_2)$. The scroll of bisecants is an R_7 which may break up into an R_5 and a K_2 or into an R_3 and $2K_2$.

65. $d_3+g_3+C_9^2$. The g_3 goes through the P_4. The C_9 has P_3' at the P_4 and at the $2P_3'$. It has $3P_1'$ on the g_3. $p'=1, 0$.

66. $d_3+g_3+C_6^2+C_3^2$. The g_3 goes through the P_4. The C_6 has P_2' at the P_4 and at the $2P_3$. It has $2P_1'$ on the g_3. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the C_6 twice and the g_3 once again.

67. $d_3+g_3+3C_3^2$. The g_3 goes through the P_4. Each C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. Each meets the g_3 and each of the other C_3 again.

68. $d_3+g_3+g_2+C_8^2$. The g_3 goes through the P_4; the g_2, through $2P_3$. The g_3 and g_2 may be consecutive. The C_8 has a P_3' at the P_4 and $3P_1'$ on the g_3. It has $2P_2'$ and a P_1' on the g_2. $p'=1, 0$.

69. $d_3+g_3+g_2+C_6^2+C_2^2$. The g_3 and the g_2 as in (68). The C_6 has a P_2' at the P_4 and at the $2P_3$. It meets the g_3 and the C_2 each in two more P_1'. The C_2 has a P_1' at the P_4, meets the g_3 again and has a P_1' on the g_2.

70. $d_3+g_3+g_2+C_5^2+C_3^2$. The g_3 and the g_2 as in (68). The C_5 has a P_2' at the P_4, $2P_1'$ on the g_3 and $3P_1'$ on the g_2. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the C_5 twice, and the g_3 once, again.

71. $d_3+g_3+g_2+2C_3^2+C_2^2$. The g_3 and the g_2 as in (68). The C_2 has a P_1' at the P_4, meets the g_3 again and has a P_1' on the g_2. Each C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. Each meets the g_3, the C_2 and the other C_3 once again.

72. $d_3+g_3+(2g_2)+C_7^2$. The g_3 goes through the P_4; the $(2g_2)$ through $2P_3$. The C_7 has a P_3' at the P_4, $3P_1'$ on the g_3 and $4P_1'$ on the $(2g_2)$. $p'=0$.

73. $d_3+g_3+(2g_2)+C_5^2+C_2^2$. The g_3 and the $(2g_2)$ as in (72). The C_5 has a P_2' at the P_4 and meets the g_3 twice, and the C_2 once, again. It has $3P_1'$ on the $(2g_2)$. The C_2 has a P_1' at the P_4 and meets the g_3 again. It has a P_1' on the $(2g_2)$.

74. $d_3+g_3+(2g_2)+C_3^2+2C_2^2$. The g_3 and the $(2g_2)$ as in (72). Each C_2 goes through the P_4, meets the g_3 again and has a P_1' on the $(2g_2)$. The C_3 is gauche, has P_1' at the P_4 and at the $2P_3$. It meets the g_3 and each C_2 once again.

Seven Threefold Points.

89. The R_7 is transformed by I into a C_5 meeting $\bar{x}=\bar{y}=0$ and passing through each fundamental point. The scroll of bisecants is an R_{12}. Taking temporarily a trisecant of the C_5 for $\bar{y}=\bar{z}=0$ in transformation II, the C_5 is transformed into an R_6 with a d_2. From the known properties of the nodal curve of such an R_6,* it is seen that the R_{12} of bisecants may break up into an R_8 and an R_4 or into $3R_4$. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 points besides the fundamental points.

1. $d_3+C^2_{12}$; $(d_2+g_1)+C^2_{12}$; $(d_1+2g_1)+C^2_{12}$. The C_{12} has P'_3 at the $7P_3$. $p'=1, 0$.

2. $d_3+C^2_8+C^2_4$; $(d_2+g_1)+C^2_8+C^2_4$; $(d_1+2g_1)+C^2_8+C^2_4$. The C_8 has P'_2 at the $7P_3$. The C_4 has P'_1 at the $7P_3$ and meets the C_8 again in $2P'_1$.

3. $d_3+3C^2_4$; $(d_2+g_1)+3C^2_4$; $(d_1+2g_1)+3C^2_4$. Each C_4 has P'_1 at the $7P_3$ and meets each of the other C_4 again in a P'_1.

90. When the R_7 has a g_2 it transforms into a C_5 with a P'_2. The scroll of bisecants is an R_{11}. In the same way as before it may be seen that the R_{11} may break up into an R_8 and an R_3 or into an R_7 and an R_4 or into $2R_4$ and an R_3. $\bar{z}=\bar{w}=0$ may meet the C_5, in addition to the fundamental points, in a P'_1 or at the P'_2.

91. When the tangents at the P'_2 cut $\bar{x}=\bar{y}=0$, the multiple generator is double torsal. The scroll of bisecants is an R_{10} which may break up into an R_7 and an R_3 or into an R_4 and $2R_3$.

4. $d_3+g_2+C^2_{11}$; $(d_2+g_1)+g_2+C^2_{11}$; $(d_1+g_2)+C^2_{11}$. The C_{11} has P'_3 at the five discrete P_3 and $2P'_2$ and a P'_1 on the g_2. $p'=1, 0$.

5. $d_3+g_2+C^2_8+C^2_3$; $(d_2+g_1)+g_2+C^2_8+C^2_3$; $(d_1+g_2)+C^2_8+C^2_3$. The C_8 has P'_2 at the $7P_3$. The C_3 has P'_1 at the five discrete P_3 and meets the C_8 again in $2P'_1$. It has a P'_1 on the g_2.

6. $d_3+g_2+C^2_7+C^2_4$; $(d_2+g_1)+C^2_7+C^2_4$; $(d_1+g_2)+C^2_7+C^2_4$. The C_7 has P'_2 at the five discrete P_3 and $3P'_1$ on the g_2. The C_4 has P'_1 at the $7P_3$ and meets the C_7 again in $2P'_1$.

7. $d_3+g_2+2C^2_4+C^2_3$; $(d_2+g_1)+2C^2_4+C^2_3$; $(d_1+g_2)+2C^2_4+C^2_3$. The C_3 has P'_1 at the five discrete P_3 and a P'_1 on the g_2. Each C_4 has P'_1 at the $7P_3$. Each C_4 meets the other C_4 and the C_3 each again in a P'_1.

*See Wiman, loc. cit. p. 31. Snyder, Amer. Journal of Math., Vol. XXVII, p. 80.

8. $d_3 + (2g_2) + C^2_{10}$; $(d_2 + g_1) + C^2_{10}$; $[d_1 + (2g_2)] + C^2_{10}$. The C_{10} has P'_3 at the five discrete P_3 and $4P'_1$ on the $(2g_2)$. $p' = 0$.

9. $d_3 + (2g_2) + C^2_7 + C^2_3$; $(d_2 + g_1) + (2g_2) + C^2_7 + C^2_3$; $[d_1 + (2g_2)] + C^2_7 + C^2_3$. The C_7 has P'_2 at the five discrete P_3 and $3P'_1$ on the $(2g_2)$. The C_3 has P'_1 at the five discrete P_3 and meets the C_7 again. It has a P'_1 on the $(2g_2)$.

10. $d_3 + (2g_2) + C^2_4 + 2C^2_3$; $(d_2 + g_1) + C^2_4 + 2C^2_3$; $(d_1 + g_2) + C^2_4 + 2C^2_3$. The C_4 has P'_1 at the $7P_3$. Each C_3 has P'_1 at the five discrete P_3, a P_1 on the $(2g_2)$ and meets the C_4 again in a P'_1.

92. When the R_7 has $2g_2$, the C_5 has $2P'_2$. The scroll of bisecants is an R_{10} which, it is readily seen, may break up into an R_8 and a K_2, an R_7 and an R_3, an R_6 and an R_4, $2R_4$ and a K_2 or into an R_4 and $2R_3$. $\bar{z} = \bar{w} = 0$ may be a trisecant of the C_5 except when the scroll of bisecants has a component K_2.

93. When the tangents at one P'_2 cut $\bar{x} = \bar{y} = 0$, one multiple generator of the R_7 is a $(2g_2)$. The scroll of bisecants of the C_5 in an R_9 which may break up into an R_7 and a K_2, an R_6 and an R_3, an R_4 and an R_3 and an R_2 or into $3R_3$. $\bar{z} = \bar{w} = 0$ be a trisecant except when the scroll of bisecants has a component K_2.

94. When the tangents of both P'_2 cut $\bar{x} = \bar{y} = 0$, then both multiple generators are double torsal. The scroll of bisecants is an R_6 and a K_2 or $2R_3$ and a K_2.

11. $d_3 + 2g_2 + C^2_{10}$; $(d_2 + g_1) + 2g_2 + C^2_{10}$. The C_{10} has P'_3 at the three discrete P_3 and $2P'_2$ and a P'_1 on each g_2. $p' = 1, 0$.

12. $d_3 + 2g_2 + C^2_8 + C^2_2$. The C_8 has P'_2 at the $7P_3$. The C_2 has P'_1 at the three discrete P_3, meets the C_8 again in $2P'_1$ and meets each g_2 once.

13. $d_3 + 2g_2 + C^2_7 + C^2_3$; $(d_2 + g_1) + 2g_2 + C^2_7 + C^2_3$. The C_7 has P'_2 at the three discrete P_3 and at the $2P_3$ on one g_2. It has $3P'_1$ on the other g_2. The C_3 has P'_1 at the three discrete P_3 and at the $2P_3$ on the latter g_2. It meets the C_7 again in $2P'_1$ and meets the former g_2 once.

14. $d_3 + 2g_2 + C^2_6 + C^2_4$; $(d_2 + g_1) + 2g_2 + C^2_6 + C^2_4$. The C_6 has P'_2 at the three discrete P_3 and $3P'_1$ on each g_2. The C_4 has P'_1 at the $7P_3$ and meets the C_6 again in $2P'_1$.

15. $d_3 + 2g_2 + 2C^2_4 + C^2_2$. The C_2 has P'_1 at the three discrete P_3 and meets each g_2 once. Each C_4 has P'_1 at the $7P_3$ and meets the other C_4 and the C_2 each again.

16. $d_3 + 2g_2 + C^2_4 + 2C^2_3$; $(d_2 + g_1) + 2g_2 + 2C^2_3$. The C_4 has P'_1 at the $7P_3$. Each C_3 has P'_1 at the three discrete P_3 and at the $2P_3$ on one g_2 and meets the

other g_2 in a P_1'. Each two of the curves C_4, C_3 and C_3 have another P_1' common.

17. $d_3 + (2g_2) + g_2 + C_9^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_9^2$. The C_9 has P_3' at the three discrete P_3, $2P_2'$ and a P_1' on the g_2 and $4P_1'$ on the $(2g_2)$. $p' = 0$.

18. $d_3 + (2g_2) + g_2 + C_7^2 + C_2^2$. The C_7 has P_2' at the three discrete P_3, $2P_2'$ on the g_2 and $3P_1'$ on the $(2g_2)$. The C_2 has P_1' at the three discrete P_3, meets the C_7 in another P_1', has a P_1' on the g_2 and one on the $(2g_2)$.

19. $d_3 + (2g_2) + g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the three discrete P_3, $3P_1'$ on the g_2 and $3P_1'$ on the $(2g_2)$. The C_3 goes through the three discrete P_3 and the $2P_3$ on the g_2. It has another P_1' on the C_6 and a P_1' on the $(2g_2)$.

20. $d_3 + (2g_2) + g_2 + C_4^2 + C_3^2 + C_2^2$. The C_4 has P_1' at the $7P_3$ and meets the C_3 and the C_2 each again. The C_3 has P_1' at the three discrete P_3 and at the $2P_3$ on the g_2. It has a P_1' on the $(2g_2)$. The C_2 has P_1' at the three discrete P_3, and P_1' on the g_2 and on the $(2g_2)$.

21. $d_3 + (2g_2) + g_2 + 3C_3^2$; $(d_2 + g_1) + 2g_2 + g_2 + 3C_3^2$. Each C_3 has P_1' at the three discrete P_3. One C_3 has $2P_1'$ on the $(2g_2)$, a P_1' on the g_2 and another P_1' on each of the other C_3. Each of the other C_3 has $2P_1'$ on the g_2 and one P_1' on the $2g_2$.

22. $d_3 + 2(2g_2) + C_6^2 + C_2^2$. The C_6 has P_2' at the three discrete P_3 and $3P_1'$ on each $(2g_2)$. The C_2 has P_1' at the three discrete P_3 and has a P_1' on each $(2g_2)$.

23. $d_3 + 2(2g_2) + 2C_3^2 + C_2^2$. The $2C_3$ and C_2 each have P_1' at the three discrete P_3. Each C_3 has $2P_1'$ on one $(2g_2)$, and one P_1' on the other $(2g_2)$ and meets the other C_3 again. The C_2 has a P_1' on each $(2g_2)$.

95. When the R_7 has $3g_2$, take one g_2 for $y = z = 0$ in transformation II. The R_7 is transformed into a C_5 with $2P_2'$. The C_5 meets $\bar{x} = \bar{y} = 0$ once and $\bar{y} = \bar{z} = 0$ twice. We have seen that the scroll of bisecants to such a C_5 is an R_{10} which may break up into an R_8 and a K_2, an R_7 and an R_3, an R_6 and an R_4, $2R_4$ and a K_2 or into an R_4 and $2R_3$. $\bar{y} = \bar{z} = 0$ is a simple generator of the scroll of bisecants. When the C_5 passes through $\bar{x} = \bar{y} = \bar{z} = 0$, the R_7 has a g_1 coinciding with the directrix. One, two, or all three multiple generators may be double torsals. In the latter case the tangents at both double points meet $\bar{x} = \bar{y} = 0$ and the C_5 touches $\bar{y} = 0$ twice on $\bar{y} = \bar{z} = 0$.

96. Whenever the scroll of bisecants has a component K_2 of which $\bar{y} = \bar{z} = 0$ is a generator, the R_7 has a double contact directrix $(\delta_{2,1} + g_1)$. To the two points on an arbitrary generator of the K_2 corresponds two generators intersecting on $\bar{x} = \bar{y} = 0$ and coplanar with $\bar{x} = \bar{y} = 0$. The directrix counts once as a generator.

24. $d_3 + 3g_2 + C_9^2$; $(d_2 + g_1) + 3g_2 + C_9^2$. The C_9 has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on each g_2. $p' = 1, 0$.

25. $d_3 + 3g_2 + C_7^2 + C_2^2$. The C_7 has P_2' at the discrete P_3 and at both P_3 on $2g_2$. It meets the other g_2 in $3P_1'$. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the latter g_2. It meets each of the other g_2 once and meets the C_7 in two other P_1'.

26. $d_3 + 3g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + 3g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the discrete P_3 and at the $2P_3$ on one g_2 and $3P_1'$ on each of the other g_2. The C_3 has P_1' at the $2P_3$ on each of the latter $2g_2$, a P_1' on the former g_2, a P_1' at the discrete P_3 and meets the C_6 again in $2P_1'$.

27. $d_3 + 3g_2 + C_5^2 + C_4^2$; $(d_2 + g_1) + 3g_2 + C_5^2 + C_4^2$. The C_5 has a P_2' at at the discrete P_3 and $3P_1'$ on each g_2. The C_4 has P_1' at the $7P_3$ and two other P_1' on the C_5.

28. $d_3 + 3g_2 + C_4^2 + C_3^2 + C_2^2$. The C_4 has P_1' on the $7P_3$. The C_3 has P_1' at the discrete P_3 and at both P_3 on $2g_2$ and meets the other g_2 once. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the latter g_2. It meets each of the other g_2 once. Each two of the curves C_4, C_3, and C_2 have another point in common.

29. $d_3 + 3g_2 + 3C_3^2$; $(d_2 + g_1) + 3g_2 + 3C_3^2$. Each C_3 has P_1' at the discrete P_3 and at both P_3 on $2g_2$, meets the other g_2 once again and has another P_1' on each of the other C_3.

30. $(\delta_{2,1} + g_1) + 3g_2 + C_8^2$. The C_8 has P_2' at the $7P_3$.

31. $(\delta_{2,1} + g_1) + 3g_2 + 2C_4^2$. Each C_4 has P_1' at the $7P_3$ and another P_1' on the other C_4.

32. $d_3 + (2g_2) + 2g_2 + C_8^2$; $(d_2 + g_1) + (2g_2) + 2g_2 + C_8^2$. The C_8 has a P_3' at the discrete P_2, $2P_2'$ and a P_1' on each g_2 and $4P_1'$ on the $(2g_2)$. $p' = 0$.

33. $d_3 + (2g_2) + 2g_2 + C_6^2 + C_2^2$. The C_6 has P_2' at the discrete P_3, and at the $2P_3$ on one g_2. It has $3P_1'$ on the other g_2 and on the $(2g_2)$. The C_2 has a P_1' at the discrete P_3, $2P_1'$ on the second g_2, a P_1' on the former g_2 and on the $(2g_2)$ and another P_1' on the C_6.

34. $d_3 + (2g_2) + 2g_2 + C_5^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + 2g_2 + C_5^2 + C_3^2$. The C_5 has a P_2' at the discrete P_3, P_1' at the other $6P_3$ and meets the C_3, the $2g_2$ and the $(2g_2)$ each again. The C_3 has P_1' at the $7P_3$.

35. $d_3 + (2g_2) + 2g_2 + C_4^2 + 2C_2^2$. The C_4 has P_1' at the $7P_3$. Each C_2 has P_1' at the discrete P_3 and at the $2P_3$ on one g_2, meets the other g_2 and the $(2g_2)$ each once and has another P_1' on the C_4.

36. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. One C_3 has P_1' at the discrete P_3, at the $2P_3$ on the $(2g_2)$ and at the $2P_3$ on one g_2. It has a P_1' on the other g_2 and meets the C_2 and the other C_3 each again. The other C_3 has P_1' at the discrete P_3 and at the $2P_3$ on each g_2 and has a P_1' on the $(2g_2)$. The C_2 has P_1' at the discrete P_3 and at the $2P_2'$ on the second g_2. It has a P_1' on the g_2 and one on the $(2g_2)$.

37. $d_3 + (2g_2) + 2g_2 + 2C_3^2 + C_2^2$. Each C_3 has P_1' at the discrete P_3 and at the $2P_3$ on each g_2. Each has a P_1' on the $(2g_2)$. The C_2 has a P_1' at the discrete P_3 and meets each C_3 again. It meets the $(2g_2)$ twice and each g_2 once.

38. $(\delta_{2,1} + g_1) + (2g_2) + 2g_2 + C_7^2$. The C_7 has P_2' at $5P_3$, P_1' at the $2P_3$ on the $(2g_2)$. It meets the $(2g_2)$ once again.

39. $(\delta_{2,1} + g_1) + (2g_2) + 2g_2 + C_4^2 + C_3^2$. The C_4 has P_1' at the $7P_3$. The C_3 has P_1' at $5P_3$, meets the C_4 again and has a P_1' on the $(2g_2)$.

40. $d_3 + 2(2g_2) + g_2 + C_5^2 + C_2^2$. The C_5 has a P_2' at the discrete P_3 and P_1' at the other $6P_3$. It meets the g_2 and each $(2g_2)$ again. The C_2 has P_1' at the discrete C_3 and at the $2P_3$ on the g_2. It has a P_1' on each $(2g_2)$.

41. $d_3 + 2(2g_2) + g_2 + C_3^2 + 2C_2^2$. The C_3 has P_1' at the discrete P_3, at the $2P_3$ on the g_2 and at those on one $(2g_2)$. It has a P_1' on the other $(2g_2)$. One C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the second $(2g_2)$. It meets the C_3 again, has a P_1' on the $(2g_2)$ and one on the g_2. The other C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the g_2. It has a P_1' on each $(2g_2)$.

42. $(\delta_{2,1} + g_1) + 2(2g_2) + g_2 + C_6^2$. The C_6 has P_2' at the P_3 on the directrix and at the $2P_3$ on the g_2. It has $3P_1'$ on each $(2g_2)$.

43. $(\delta_{2,1} + g_1) + 2(2g_2) + g_2 + 2C_3^2$. Each C_3 has P_1' at the P_3 on the directrix and at the $2P_3$ on the g_2. Each has $2P_1'$ on one $(2g_2)$ and $1P_1'$ on the other.

44. $d_3 + 3(2g_2) + 3C_2^2$. Each C_2 goes through the discrete P_3 and the $2P_3$ on one $(2g_2)$. Each has a P_1' on each of the other $(2g_2)$.

$p = 1.$

Fourfold Point.

97. The R_7 is transformed by I into a C_4 of the first kind passing through one fundamental point. The scroll of bisecants is an R_8 of genus 3, 2, 1. It may break up into an R_6 and a K_2 or into an R_4 and $2K_2$.

98. When the fundamental point not on the C_4 lies on the scroll of bisecants, the R_7 has a g_2 through the P_4. When it lies at the intersection of two generators of the scroll of bisecants, the R_7 has $2g_2$ through the P_4. When the fundamental point lies on a $(2g_2)$ of the scroll of bisecants the R_7 has a $(2g_2)$, or, in particular, if it lies at a pinch-point, a $(3g_2)$, through the P_4.

The R_7 cannot have a g_2 not passing through the P_4.

1. $d_3 + C_{11}^2$; $(d_2 + g_1) + C_{11}^2$. The C_{11} has a P_6' at the P_4 and a P_3' at the P_3. $p' = 3,\ 2,\ 1$.

2. $d_3 + C_8^2 + C_3^2$; $(d_2 + g_1) + C_8^2 + C_3^2$. The C_8 has a P_4' at the P_4 and a P_2' at the P_3. p' for the C_8 is 2 or 1. The C_3 has a P_2' at the P_4, a P_1' at the P_3 and two other P_1' on the C_8.

3. $d_3 + C_5^2 + 2C_3^2$; $(d_2 + g_1) + C_5^2 + 2C_3^2$. Each curve has a P_2' at the P_4 and a P_1' at the P_3. The C_5 meets each C_3 again in $2P_1'$. p' for the C_5 is 1.

4. $d_3 + g_2 + C_{10}^2$; $(d_2 + g_1) + g_2 + C_{10}^2$. The C_{10} has a P_5' at the P_4 and a P_3' at the P_3. It meets the g_2 again in a P_1'. $p' = 3,\ 2$, or 1.

5. $d_3 + g_2 + C_7^2 + C_3^2$; $(d_2 + g_1) + g_2 + C_7^2 + C_3^2$. The C_7 has a P_3' at the P_4 and a P_2' at the P_3. It meets the g_2 again. $p' = 2,\ 1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3, and two other P_1' on the C_7.

6. $d_3 + g_2 + C_8^2 + C_2^2$. The C_8 has a P_4' at the P_4 and a P_2' at the P_3. $p' = 2$ or 1. The C_2 has P_1' at the P_4 and at the P_3. It meets the C_8 twice and the g_2 once again.

7. $d_3 + g_2 + C_4^2 + 2C_3^2$; $(d_2 + g_1) + g_2 + C_4^2 + 2C_3^2$. The C_4 has P_1' at the P_4 and at the P_3. It meets the g_2 again. $p' = 1$. Each C_3 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_4 twice again.

8. $d_3 + g_2 + C_5^2 + C_3^2 + C_2^2$. The C_5 and C_3 each have a P_2' at the P_4 and a P_1' at the P_3. The C_2 has P_1' at the P_4' and at the P_3 and meets the g_2 again. The C_5 meets the C_3 and C_2 each twice again. Its genus is 1.

9. $d_3 + 2g_2 + C_9^2$; $(d_2 + g_1) + 2g_2 + C_9^2$. The C_9 has a P_4' at the P_4, and a P_3' at the P_3. It meets each g_2 again. $p' = 3,\ 2,\ 1$.

10. $d_3 + 2g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + 2g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the P_4 and P_3, and meets each g_2 again. $p' = 2$ or 1. The C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the C_6 twice again.

11. $d_3 + 2g_2 + 3C_3^2$; $(d_2 + g_1) + 2g_2 + 3C_3^2$. One C_3 has a P_1' at the P_3 and one on each g_2. Its genus is 1. Each of the other two C_3 has a P_2' at the P_4, a P_1' at the P_3 and two other P_1' on the first C_3.

12. $d_3 + (2g_2) + C_9^2$. The C_9 has at the P_4 a P_4' with two branches touching the $(2g_2)$. It has a P_3' at the P_3. $p' = 2$ or 1.

13. $d_3 + (2g_2) + C_6^2 + C_3^2$. The C_6 has two consecutive P_2' at the P_4 and a P_2' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, and a P_1' at the P_3. It meets the C_6 again in $2P_1'$.

14. $d_3 + (2g_2) + C_7^2 + C_2^2$. The C_7 has at the P_4 a P_3' with one branch touching the $(2g_2)$ and a P_2' at the P_3. $p' = 2$ or 1. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the P_3 and meets the C_7 once again.

15. $d_3 + (2g_2) + C_4^2 + C_3^2 + C_2^2$. The C_4 touches the $(2g_2)$ at the P_4 and has a P_1' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the C_4 twice again. The C_2 touches the $(2g_2)$ at the P_4, has a P_1' at the P_3, and meets the C_4 once again.

16. $d_3 + (3g_2) + C_8^2$; $(d_2 + g_1) + (3g_2) + C_8^2$. The C_8 has, at the P_4, a P_3' with one branch touching the $(3g_2)$. It has a P_3' at the P_3 and meets the $(3g_2)$ again. $p' = 2$, or 1.

17. $d_3 + (3g_2) + C_5^2 + C_3^2$; $(d_2 + g_1) + (3g_2) + C_5^2 + C_3^2$. The C_5 touches the $(3g_2)$ at the P_4, meets the $(3g_3)$ again, and has a P_2' at the P_3. $p' = 1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_5 twice again.

18. $d_3 + (3g_2) + C_6^2 + C_2^2$. The C_6 has P_2' at the P_4 and at the P_3 and meets the $(3g_2)$ again. $p' = 2$ or 1. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the P_3, and meets the C_6 once again.

19. $d_3 + (3g_2) + 2C_3^2 + C_2^2$. One C_3 has a P_1' at the P_3 and a P_1' on the $(3g_2)$. $p' = 1$. The other C_3 has a P_2' at the P_4, a P_1' at the P_3, and meets the other C_3 in $2P_1'$. The C_2 touches the $(3g_2)$ at the P_4, has a P_1' at the P_3, and meets the first C_3 again.

99. When the R_7 has a double contact directrix, the P_3 is on the directrix. Taking the P_4 and a P_2 for fundamental points the R_7 is transformed into a C_5 with a P_2' at one fundamental point and lying on an R_3 which has $\bar{x} = \bar{y} = 0$ for d_1 and $\bar{z} = \bar{w} = 0$ for d_2. The residual scroll of bisecants is an R_7 of genus 2, or 1,

or an R_4 of genus 1 and an R_3 or an R_5 of genus 1 and a K_2. The fundamental point not on the C_5 may be at a pinch-point of a $(2g_2)$ of the scroll of bisecants, giving rise to a $(3g_2)$.

20. $(\delta_{2,1}+g_1)+2g_2+C_8^2$. The C_8 has a P_4' at the P_4 and a P_2' at the P_3. $p'=2$ or 1.

21. $(\delta_{2,1}+g_1)+2g_2+C_5^2+C_3^2$. The C_5 and C_3 each have a P_2' at the P_4 and a P_1' at the P_3. They meet again in $2P_1'$. p' for the C_5 is 1.

22. $(\delta_{2,1}+g_1)+(3g_2)+C_7^2$. The C_7 has, at the P_4, a P_3' with one branch touching the $(3g_2)$. It has a P_2' at the P_3. $p'=2$ or 1.

23. $(\delta_{2,1}+g_1)+(3g_2)+C_4^2+C_3^2$. The C_4 touches the $(3g_2)$ at the P_4 and has a P_1' at the P_3. $p'=1$. The C_3 has a P_2' at the P_4, a P_1' at the P_3 and meets the C_4 twice again.

100. When the R_7 has a g_3 it may be transformed by II into a C_4 of the first kind cutting $\bar{y}=\bar{z}=0$ once. We have seen that the scroll of the bisecants is an R_8 which may break up into an R_6 and a K_2 or into an R_4 and $2K_2$.

24. $d_3+g_3+C_8^2$. The C_8 has a P_3' at the P_4 and $3P_1'$ on the g_3. $p'=3$, 2, or 1.

25. $d_3+g_3+C_6^2+C_2^2$. The C_6 has a P_2' at the P_4 and $2P_1'$ on the g_3. $p'=2$ or 1. The C_2 has a P_1' at the P_4 and meets the C_6 twice, and the g_3 once again.

26. $d_3+g_3+C_4^2+2C_2^2$. Each curve has a P_1' at the P_4 and meets the g_3 again. The C_4 meets each C_2 twice again. p' for the C_4 is 1.

Five Threefold Points.

101. The R_7 is transformed by I into a C_5 of genus 1 meeting $\bar{x}=\bar{y}=0$ and passing through both fundamental points. Taking, temporarily, a trisecant cutting $\bar{x}=\bar{y}=0$ for $\bar{y}=\bar{z}=0$ in transformation II, the C_5 may be transformed into an R_6 with a d_2. From known properties of the nodal curve of such of R_6,* it follows that the R_{11} of bisecants to the C_5 may break up into an R_8 and an R_3 or into an R_5 and $2R_3$. $\bar{z}=\bar{w}=0$ may meet the C_5 in one point not a fundamental point.

102. When the R_7 has a g_2 the C_5 has a P_2'. The scroll of bisecants is an R_{10} which may break up into an R_7 and an R_3, into an R_8 and a K_2, into an R_4 and $2R_3$ or into an R_5, an R_3 and a K_2.

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*See Wiman loc. cit. p. 52. Snyder, Amer. Journal of Math. Vol. 25, p. 96.

103. When the multiple generator is a $(2g_2)$ the tangents at the P_2' cut $\bar{x} = \bar{y} = 0$. The scroll of bisecants is an R_9 which may break up into an R_6 and an R_3, an R_7 and a K_2, or an R_4, an R_3, and a K_2.

104. When the R_7 has $2g_2$, take one of them for $y = z = 0$ in transformation II. The R_7 goes into a C_5 with a P_2' as obtained for the transform of an R_7 with one g_2. It meets $\bar{x} = \bar{y} = 0$ once and $\bar{y} = \bar{z} = 0$ twice. When the tangents at the P_2' meet $\bar{x} = \bar{y} = 0$ one multiple generator is double torsal. When in addition $\bar{y} = \bar{z} = 0$ joins the points of tangency of two tangents lying in $\bar{y} = 0$, both multiple generators are double torsal.

105. When the R_7 has two multiple generators, it may have a double contact directrix. This happens when the scroll of bisecants has a component K_2 on which $\bar{y} = \bar{z} = 0$ is a generator.

1. $d_3 + C_{11}^2$; $(d_2 + g_1) + C_{11}^2$. The C_{11} has P_3' at the $5P_3$. $p' = 3$, 2, or 1.

2. $d_3 + C_8^2 + C_3^2$; $(d_2 + g_1) + C_8^2 + C_3^2$. The C_8 has P_2' at the $5P_3$. $p' = 2$ or 1. The C_3 has P_1' at the $5P_3$ and meets the C_8 again in $2P_1'$.

3. $d_3 + C_5^2 + 2C_3^2$; $(d_2 + g_1) + C_5^2 + 2C_3^2$. All three curves have P_1' at the $5P_3$. The C_5 meets each C_3 again in $2P_1'$ and is of genus 1.

4. $d_3 + g_2 + C_{10}^2$; $(d_2 + g_1) + g_2 + C_{10}^2$. The C_{10} has P_3' at the three discrete P_3 and $2P_2'$ and a P_1' on the g_2. $p' = 3$, 2, or 1.

5. $d_3 + g_2 + C_8^2 + C_2^2$. The C_8 has P_2' at the $5P_3$. $p' = 2$ or 1. The C_2 has P_1 at the three discrete P_3, meets the C_8 twice again and meets the g_2 once.

6. $d_3 + g_2 + C_7^2 + C_3^2$; $(d_2 + g_1) + g_2 + C_7^2 + C_3^2$. The C_7 has P_2' at the three discrete P_3 and $3P_1'$ on the g_2. $p' = 2$ or 1. The C_3 has P_1' at the $5P_3$ and meets the C_7 twice again.

7. $d_3 + g_2 + C_4^2 + 2C_3^2$; $(d_2 + g_1) + g_2 + C_4^2 + 2C_3^2$. The C_4 has P_1' at the three discrete P_3 and a P_1' on the g_2. $p' = 1$. Each C_3 has P_1' at the $5P_3$ and meets the C_4 twice again.

8. $d_3 + g_2 + C_5^2 + C_3^2 + C_2^2$. The C_5 has P_1' at the $5P_3$ and meets the C_3 and the C_2 each twice again. $p' = 1$. The C_3 has P_1' at the $5P_3$. The C_2 has P_1' at the three discrete P_3 and has a P_1' on the g_2.

9. $d_3 + (2g_2) + C_9^2$; $(d_2 + g_1) + (2g_2) + C_9^2$. The C_9 has P_3' at the three discrete P_3 and $4P_1'$ on the $(2g_2)$. $p' = 2$ or 1.

10. $d_3 + (2g_2) + C_7^2 + C_2^2$. The C_7 has P_2' at the three discrete P_3 and

$3P_1'$ on the $(2g_2)$. $p'=2$ or 1. The C_2 has P_1' at the three discrete P_3, meets the C_7 once again and meets the $(2g_2)$ once.

11. $d_3+(2g_2)+C_6^2+C_3^2$: $(d_2+g_1)+(2g_2)+C_6^2+C_3^2$. The C_6 has P_1' at the three discrete P_3 and meets the $(2g_2)$ twice. $p'=1$. The C_3 has P_1' at the $5P_3$ and meets the C_6 twice again.

12. $d_3+(2g_2)+C_4^2+C_3^2+C_2^2$. Each curve has P_1' at the $3P_3$. The C_3 goes through the pinch-points of the $(2g_2)$. The C_4 and C_2 each meet it once. The C_4 meets the C_3 twice and the C_2 once again. p' for the C_4 is one.

13. $d_3+2g_2+C_9^2$; $(d_2+g_1)+2g_2+C_9^2$. The C_9 has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on each g_2. $p'=3$, 2 or 1.

14. $d_3+2g_2+C_7^2+C_2^2$. The C_7 has P_2' at the discrete P_3 and at the $2P_3$ on one g_2. It has $3P_1'$ on the other g_2. $p'=2$ or 1. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the second g_2. It meets the C_7 in two other P_1' and meets the first g_2 once.

15. $d_3+2g_2+C_6^2+C_3^2$. The C_6 has a P_2' at the discrete P_3 and $3P_1'$ on each g_2. $p'=2$ or 1. The C_3 has P_1' at the $5P_3$ and meets the C_6 twice again.

16. $d_3+2g_2+C_5^2+2C_2^2$. The C_5 has P_1' at the $5P_3$. $p'=1$. Each C_2 has a P_1' at the discrete P_3 and at the $2P_3$ on one g_2. Each meets the C_5 twice again and meets the other g_2 once.

17. $d_3+2g_2+C_4^2+C_3^2+C_2^2$. Each curve has a P_1' at the discrete P_3. The C_4 and C_2 each have $2P_1'$ on one g_2 and one P_1' on the other. The C_3 has $2P_1'$ on each g_2. The C_4 meets the C_3 and C_2 each twice again. p' for the C_4 is 1.

18. $d_3+2g_2+3C_3^2$; $(d_2+g_1)+2g_2+3C_3^2$. Each C_3 has a P_1' at the discrete P_3. One C_3 meets each g_2 once and is of genus 1. Each of the other C_3 is gauche, has $2P_1'$ on each g_2, and meets the first C_3 twice again.

19. $(\delta_{2,1}+g_1)+2g_2+C_8^2$. The C_8 has P_2' at the $5P_3$. $p'=2$ or 1.

20. $(\delta_{2,1}+g_1)+2g_2+C_5^2+C_3^2$. The C_5 and C_3 each have P_1' at the $5P$ and meet again in $2P_1'$. The C_5 is of genus 1. The C_3 is gauche.

21. $d_3+(2g_2)+g_2+C_8^2$; $(d_2+g_1)+(2g_2)+g_2+C_8^2$. The C_8 has a P_3' at the discrete P_3, $2P_2'$ and a P_1' on the g_2 and $4P_1'$ on the $(2g_2)$. $p'=2$ or 1.

22. $d_3+(2g_2)+g_2+C_6^2+C_2^2$. The C_6 has a P_2' at the discrete P_3 and $3P_1'$ each on the g_2 and the $(2g_2)$. $p'=2$ or 1. The C_2 has P_1' at the discrete P_3 and at the $2P_3$ on the g_2, meets the C_6 once again and meets the $(2g_2)$ once.

23. $d_3+(2g_2)+g_2+C_6^2+C_2^2$. The C_6 has P_2' at the discrete P_3 and at

the $2P_3$ on the g_2. It meets the $(2g_2)$ twice. $p' = 1$. The C_2 has P_1' at the discrete P_3 and the $2P_3$ on the $(2g_2)$. It meets the C_6 twice again.

24. $d_3 + (2g_2) + g_2 + C_5^2 + C_3^2$; $(d_2 + g_1) + (2g_2) + g_2 + C_5^2 + C_3^2$. The C_5 has a P_1' at the discrete P_3, $3P_1'$ on the g_2 and $2P_1'$ on the $(2g_2)$. $p' = 1$. The C_3 is gauche. It has P_1' at the $5P_3$ and meets the C_5 twice again.

25. $d_3 + (2g_2) + g_2 + C_4^2 + 2C_2^2$. Each curve has a P_1' at the discrete P_3. One C_2 has $2P_1'$ on the g_2 and one P_1' on the $(2g_2)$. The other has a P_1' on the g_2 and $2P_1'$ on the $(2g_2)$. The C_4 has $2P_1'$ on the g_2, one P_1' on the $(2g_2)$, and meets each C_2 twice again. Its genus is 1.

26. $d_3 + (2g_2) + g_2 + 2C_3^2 + C_2^2$. Each curve has a P_1' at the discrete P_3. The C_2 has $2P_1'$ on the g_2 and one P_1' on the $(2g_2)$. One C_3 is gauche. It has $2P_1'$ each on the g_2 and the $(2g_2)$. The other C_3 is of genus 1. It meets the g_2 and the $(2g_2)$ each once, meets the C_3 twice and the C_2 once again.

27. $(\delta_{2,1} + g_1) + (2g_2) + g_2 + C_7^2$. The C_7 has P_2' at the P_3' on the directrix and the $2P_3'$ on the g_2. It has $3P_1'$ on the $(2g_2)$. $p' = 2$ or 1.

28. $(\delta_{2,1} + g_1) + (2g_2) + g_2 + C_4^2 + C_3^2$. Each curve has P_1' at the P_3 on the directrix and the $2P_3$ on the g_2. Each meets the $(2g_2)$ twice. They meet again in $2P_1'$. p' for the C_4 is 1. The C_3 is gauche.

29. $d_3 + 2(2g_2) + C_7^2$. The C_7 has a P_3' at the discrete P_3 and $4P_1'$ on each $(2g_2)$. $p' = 1$.

30. $d_3 + 2(2g_2) + C_5^2 + C_2^2$. The C_5 has a P_2' at the discrete P_3. It has $2P_1'$ on one $(2g_2)$ and $3P_1'$ on the other. $p' = 1$. The C_2 has $2P_1'$ on one $(2g_2)$ and one P_1' on the other. It meets the C_5 once again.

31. $d_3 + 2(2g_2) + C_3^2 + 2C_2^2$. Each curve has a P_1' at the discrete P_3. The C_3 has a P_1' on each $(2g_2)$. $p' = 1$. Each C_2 has $2P_1'$ on one $(2g_2)$ and one P_1' on the other $(2g_2)$. Each meets the C_3 once again.

32. $(\delta_{2,1} + g_1) + 2(2g_2) + C_6^2$. The C_6 has a P_2' at the P_3 on the directrix and $3P_1'$ on each $(2g_2)$. $p' = 2$ or 1.

33. $(\delta_{2,1} + g_1) + 2(2g_2) + 2C_3^2$. Each C_3 has a P_1' at the P_3 on the directrix. One is of genus 1 and meets each $(2g_2)$ once. The other C_3 is gauche, goes through the pinch points of each $(2g_2)$ and meets the plane C_3 twice again.

$$p = 2$$

106. The R_7 may be transformed by I into a C_5 of genus two, which meets $\bar{x} = \bar{y} = 0$ once and passes simply through each fundamental point. The C_5 has

a trisecant meeting $\bar{x} = \bar{y} = 0$ and may therefore be transformed by II into an R_6.* Hence we find that the R_{10} of bisecants to the C_5 is of genus 5, 4, 3 or 2 and may break up into an R_8 of genus 4, 3 or 2, and a K_2 into an R_6 and an R_4 each of genus 1 or into $2R_4$ of genus 1 and a K_2.

107. When the R_7 has a g_2, it transforms by II into a C_5 of the same kind as above. When the multiple generator is a $(2g_2)$ then $\bar{y} = \bar{z} = 0$ joins the points of tangency of two tangents lying in $\bar{y} = 0$.

1. $d_3 + C_{10}^2$; $(d_2 + g_1) + C_{10}^2$. The C_{10} has P_3' at the $3P_3$. $p' = 5, 4, 3$ or 2.

2. $d_3 + C_8^2 + C_2^2$. The C_8 has P_2' at the $3P_3$. $p' = 4$, 3 or 2. The C_2 has P_1' at the $3P_3$ and meets the C_8 twice again.

3. $d_3 + C_6^2 + C_4^2$; $(d_2 + g_1) + C_6^2 + C_4^2$. The C_6 has P_2' and the C_4, P_1' at the $3P_3$. They meet again in $4P_1'$. Each is of genus 1.

4. $d_3 + 2C_4^2 + C_2^2$. Each C_4 meets the C_2 once and the other C_4 thrice again. Each C_4 is of genus 1.

5. $d_3 + g_2 + C_9^2$; $(d_2 + g_1) + g_2 + C_9^2$. The C_9 has a P_3' at the discrete P_3 and $2P_2'$ and a P_1' on the g_2. $p' = 5$, 4, 3 or 2.

6. $d_3 + g_2 + C_7^2 + C_2^2$. The C_7 has a P_2' at the discrete P_3 and $3P_1'$ on the g_2. $p' = 4$, 3, or 2. The C_2 has a P_1' at the $3P_3$ and meets the C_7 again in $2P_1'$.

7. $d_3 + g_2 + C_6^2 + C_3^2$; $(d_2 + g_1) + g_2 + C_6^2 + C_3^2$. The C_6 has P_2' at the $3P_3$. The C_3 has a P_1' at the discrete P_3, meets the C_6 again in $4P_1'$ and meets the g_2. Each curve is of genus 1.

8. $d_3 + g_2 + C_5^2 + C_4^2$; $(d_2 + g_1) + g_2 + C_5^2 + C_4^2$. The C_5 has a P_2' at the discrete P_3 and $3P_1'$ on the g_2. The C_4 has P_1' at the $3P_3$ and meets the C_5 again in $4P_1'$. Each curve is of genus 1.

9. $d_3 + g_2 + C_4^2 + C_3^2 + C_2^2$. The C_4 and C_2 each have P_1' at the $3P_3$ and meet in another P_1'. The C_3 has a P_1' at the discrete P_3, meets the C_2 once and the C_4 thrice again and meets the g_2 once. $p' = 1$ for both the C_4 and the C_3.

10. $(\delta_{2,1} + g_1) + g_2 + C_8^2$. The C_8 has P_2' at the $3P_3$. $p' = 4$, 3, or 2.

11. $(\delta_{2,1} + g_1) + g_2 + 2C_4^2$. Each C_4 has P_1' at the $3P_3$ and meets the other C_4 in $3P_1'$. $p' = 1$ for each C_4.

12. $d_3 + (2g_2) + C_8^2$. The C_8 has a P_3' at the discrete P_3 and $4P_1'$ on the $(2g_2)$. $p' = 4$, 3 or 2.

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* See Wiman, *loc. cit.*, pp. 58 and 59. Snyder, *Amer. Jour. of Math.*, Vol. 25, p. 265; Vol. 27, p. 102.

13. $d_3+(2g_2)+C_6^2+C_2^2$. The C_6 has a P_2' at the discrete P_3 and $2P_1'$ on the $(2g_2)$. $p'=3$ or 2. The C_2 has P_1' at the $3P_3$ and meets the C_6 again in $2P_1'$.

14. $d_3+(2g_2)+C_5^2+C_3^2$. The C_5 has a P_2' at the discrete P_3 and $3P_1'$ on the g_2. The C_3 has a P_1' at the discrete P_3 and meets the C_5 again in $3P_1'$. It meets the $(2g_2)$ once.

15. $d_3+(2g_2)+2C_3^2+C_2^2$. Each C_3 has a P_1' at the discrete P_3, touches the torsal plane of the $(2g_2)$ at a point on the $(2g_2)$ and has two P_1' on the other g_2. $p'=1$ for each C_3. The C_2 has P_1' at the $3P_3$ and meets each C_3 again in a P_1'.

16. $(\delta_{2,1}+g_1)+(2g_2)+C_7^2$. The C_7 has a P_2' at the P_3 on the directrix and $3P_1'$ on the $(2g_2)$. $p'=4$, 3, or 2.

17. $(\delta_{2,1}+g_1)+2g_2+C_4^2+C_3^2$. The C_4 has P_1' at the $3P_3$. The C_3 has a P_1' at the P_3 on the directrix and a P_1' on the $(2g_2)$. It meets the C_4 again in $3P_1'$. Each curve is a genus 1.

$$p=3.$$

108. Take a generator through the single P_3' for $y=z=0$ in transformation II. The R_7 may thus be transformed into a C_6 of genus 3, with $2P_1'$ on $\bar{x}=\bar{y}=0$ and, on $\bar{y}=\bar{z}=0$, a P_1', and a P_2' at which both tangents are not coplanar with $\bar{y}=\bar{z}=0$. The scroll of bisecants is an R_{11} having $\bar{x}=\bar{y}=0$ for d_5. Its genus is, by formula,* seven, but may reduce to three since $\bar{x}=\bar{y}=0$ may be the intersection of four planes of the double developable. Since the C_6 lies on a quadric, the R_{11} cannot have a component R_2, K_2 or R_3. It may, however, break up into an R_8 of genus 3 or 2 and a K_3 of genus 1 or into an R_7 and an R_4, either pair of forms corresponding to a C_6^2 and a C_3^2 on the R_7. The R_{11} cannot break up into an R_5 and an R_6 for one of them would have to have the C_6 as double curve. No such R_5 or R_6 exists.

109. From considerations similar to the above it is readily seen that the double curve of the R_7 can break up into three components, if at all, only as $3C_3$ each of genus 1. That it can so decompose is readily seen by Salmon's generation of scrolls as the locus of a line meeting three curves. Take for the three curves the directrix and $2C_3$ of genus 1 each meeting the directrix and

* See paragraph 9.

having three common points at two of which the tangents to each curve meet the directrix. The R_{18} thus formed, consists of the plane pencils to the intersections of the two curves and the directrix, two of which count twice, the perspective cones of each C_3 from the intersection of the other C_3 with the directrix, and an R_7 with a d_3 having the $2C_3$ for double curves. If the directrix is taken so as not to be coplanar with any other pair of tangents of the cubics, then the R_7 must be of genus 3 for no such R_7 of lower genus has two such C_3^2. The scroll of bisecants of the C_6 may, therefore, break up into $2R_4$ and a K_3 each of genus 1. The C_6 may pass through $\bar{x}=\bar{y}=\bar{z}=0$.

1. $d_3+C_9^2$; $(d_2+g_1)+C_9^2$. The C_9 has a P_3' at the P_3. $p'=7, 6, 5, 4$, or 3.

2. $d_3+C_6^2+C_3^2$; $(d_2+g_1)+C_6^2+C_3^2$. The C_6 has a P_2' at the P_3. $p'=3$ or 2. The C_3 has a P_1' at the P_3 and meets the C_7 in $4P_1'$. Its genus is 1.

3. $d_3+3C_3^2$; $(d_2+g_1)+3C_3^2$. Each C_3 has a P_1' at the P_3 and meets each of the other C_3 in $2P_1'$. Each is of genus 1.

110. When the R_7 has a double contact directrix the P_3 is on the directrix. The R_7 is transformed by II into a C_6 with a P_2' at $\bar{x}=\bar{y}=\bar{z}=0$ at which one branch touches $y=0$. The C_6 lies on a K_2 having $\bar{y}=\bar{z}=0$ as a generator. The residual scroll of bisecants is an R_{10} having $y=z=0$ for double generator. Its genus is, in general, six but may reduce to three since $x=y=0$ may lie in three planes of the double developable of the C_6. The R_{10} may break up into an R_7 of genus 2 and a K_3 of genus 1 or into an R_6 of genus 2 and an R_4 of genus 1.

4. $(\delta_{2,1}+g_1)+C_8^2$. The C_8 has a P_2' at the P_3. $p'=6, 5, 4$, or 3.

5. $(\delta_{2,1}+g_1)+C_5^2+C_3^2$. The C_5 and C_3 each have a P_1' at the P_3 and meet in four other P_1'. The C_5 is of genus 2, the C_3 of genus 1.

Directrix a Four-fold Line on the Surface.

$$p=0$$

Triple Curves or Triple Generator.

111. When the R_7 has a C_3^3, the surface transforms into a C_4 having $x=y=0$ as directrix of the R_2 of trisicants on which the C_4 lies. It meets $\bar{x}=\bar{y}=0$ once. $\bar{z}=\bar{w}=0$ meets the C_4 in 0, 1, or 2 points.

112. When the R_7 has a C_2^3, the surface transforms into a plane C_4 with one point on $\bar{x}=\bar{y}=0$. It has $3P_2'$ or a P_3'. $\bar{z}=\bar{w}=0$ either does not meet the curve or meets it in a simple, a double, or a triple point.

113. In general, when the R_7 has a g_3, it transforms by II into a C_4 of the second kind having a P_1' on $\bar{x} = \bar{y} = 0$. The scroll of bisecants is an R_6. When the R_7 has also a g_2, the C_4 has a P_2'. The scroll of bisecants is an R_5. The C_4 may pass through $\bar{x} = \bar{y} = \bar{z} = 0$.

114. When the g_3 coincides with $x = y = 0$, take the discrete P_3 and a P_2 as fundamental points in transformation I. The R_7 goes into a C_5 with $2P_1'$ on $\bar{x} = \bar{y} = 0$. It passes through one fundamental point and has a P_3' on $\bar{z} = \bar{w} = 0$. The scroll of bisecants is an R_5.

1. $d_4 + C_3^3$; $(d_3 + g_1) + C_3^3$; $(d_2 + 2g_1) + C_3^3$.
2. $d_4 + 3g_2 + C_2^3$; $(d_3 + g_1) + 3g_2 + C_2^3$; $(d_2 + g_2) + 2g_2 + C_2^3$.
3. $d_4 + g_3 + C_2^3$; $(d_3 + g_1) + g_3 + C_2^3$; $(d_1 + g_3) + C_2^3$.
4. $d_4 + g_3 + C_6^2$; $(d_3 + g_1) + g_3 + C_6^2$; $(d_1 + g_3) + C_6^2$. The C_6 has a P_3' at the P_3 and $3P_1'$ on the g_3. $p' = 0$.
5. $d_4 + g_3 + g_2 + C_5^2$; $(d_3 + g_1) + g_3 + g_2 + C_5^2$. The C_5 has a P_2' and a P_1' on the g_2 and $3P_1'$ on the g_3. $p' = 0$.

Double Curves.

115. Taking $2P_3'$ for the fundamental points in transformation I, the R_7 transforms into a C_4 with a P_1' on $\bar{x} = \bar{y} = 0$. The scroll of bisecants is an R_6. It cannot break up. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1, 2 or 3 points.

116. When the R_7 has a g_2 the C_4 has a P_2'. The scroll of bisecants is an R_5. $\bar{z} = \bar{w} = 0$ meets the C_4 in 0, 1, or 2 P_1' or in the P_2' and in 0 or 1 P_1'. When one fundamental point lies on the R_5 of bisecants, the R_7 has $2g_2$. When both fundamental points lie on the C_5 then the R_7 has $3g_2$.

1. $d_4 + C_9^2$; $(d_3 + g_1) + C_9^2$; $(d_2 + 2g_1) + C_9^2$; $(d_1 + 3g_1) + C_9^2$. The C_9 has P_3' at the $3P_3$. $p' = 0$.
2. $d_4 + g_2 + C_8^2$; $(d_3 + g_1) + g_2 + C_8^2$; $(d_2 + 2g_1) + g_2 + C_8^2$; $(d_2 + g_2) + C_8^2$; $(d_1 + g_1 + g_2) + C_8^2$. The C_8 has P_3' at the two discrete P_3 and a P_2' on the g_2. $p' = 0$.
3. $d_4 + 2g_2 + C_7^2$; $(d_3 + g_1) + 2g_2 + C_7^2$; $(d_2 + 2g_1) + 2g_2 + C_7^2$; $(d_2 + g_2) + g_2 + C_7^2$. The C_7 has a P_3' at the discrete P_3 and a P_2' and a P_1' on each g_2. $p' = 0$.
4. $d_4 + 3g_2 + C_6^2$; $(d_3 + g_1) + 3g_2 + C_6^2$; $(d_2 + g_2) + 2g_2 + C_6^2$. The C_6 has a P_2' and a P_1' on each g_2. $p' = 0$.

$p=1$

Triple Curve or Triple Generator.

117. The R_7 cannot have a C_3^3 since a C_3^3 and a d_4 would give a nodal curve of order 15. When the R_7 has a C_2^3 it transforms by I into a plane C_4 with $2P_2'$. $\bar{z}=\bar{w}=0$ either does not meet the curve or meets it in a P_1' or a P_2'.

118. When the R_7 has a g_3 it transforms by II into a gauche C_4 of the first kind meeting $\bar{x}=\bar{y}=0$ in a P_1'. The scroll of bisecants is an R_5. The C_4 may pass through $\bar{x}=\bar{y}=\bar{z}=0$. The g_3 cannot coincide with the directrix of the R_7 for the latter would then be a simple rectilinear directrix which a scroll of genus one cannot have.

1. $d_4+2g_2+C_2^3$; $(d_3+g_1)+2g_2+C_2^3$; $(d_2+g_2)+g_2+C_2^3$.

2. $d_4+g_3+C_5^2$; $(d_3+g_1)+g_3+C_5^2$. The C_5 has $3P_1'$ on the g_3. $p'=1$.

Double Curves.

119. Taking the $2P_3$ for fundamental points, the R_7 transforms into a C_4 of the first kind. The scroll of bisecants is an R_5. $\bar{z}=\bar{w}=0$ meets the C_4 in 0, 1 or 2 P_1'.

120. When one fundamental point lies on the R_5 of bisecants, the R_7 has a g_2. When both fundamental points lie on the R_5 the R_7 has $2g_2$. By taking one fundamental point at a P_2 instead of at a P_3 of the R_7 it may be seen that one g_2 may coincide with the directrix.

1. $d_4+C_8^2$; $(d_3+g_1)+C_8^2$; $(d_2+2g_1)+C_8^2$. The C_8 has P_3' at the $2P_3$. $p'=1$.

2. $d_4+g_2+C_7^2$; $(d_3+g_1)+g_2+C_7^2$; $(d_2+2g_1)+g_2+C_7^2$; $(d_2+g_2)+C_7^2$. The C_7 has a P_3' at the discrete P_3 and a P_2' and a P_1' on the g_2. $p'=1$.

3. $d_4+2g_2+C_6^2$; $(d_3+g_1)+2g_2+C_6^2$; $(d_2+g_2)+g_2+C_6^2$. The C_6 has a P_2' and a P_1' on each g_2. $p'=1$.

$p=2$

Triple Curve.

121. The R_7 can have only a triple conic. By I it transforms into a plane C_4 with a P_2'. $\bar{z}=w=0$ either does not meet it or meets it in a P_1' or a P_2'.

1. $d_4+g_2+C_2^3$; $(d_3+g_1)+g_2+C_2^3$; $(d_2+g_2)+C_2^3$.

Double Curves.

122. The R_7 has just one P_3 and transforms into a C_5 meeting $\bar{x}=\bar{y}=0$ twice and passing through one fundamental point. The scroll of bisecants is an R_6. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 points besides the fundamental point.

123. When the fundamental point not on the C_5 lies on the R_6 of bisecants the R_7 has a g_2. It may, as before, be shown that the g_2 may coincide with the directrix.

1. $d_4+C_7^2$; $(d_3+g_1)+C_7^2$; $(d_2+2g_1)+C_7^2$. The C_7 has a P_3' at the P_3. $P'=2$.

2. $d_4+g_2+C_6^2$; $(d_3+g_1)+g_2+C_6^2$; $(d_2+g_2)+C_6^2$. The C_6 has a P_2' and a P_1' on the g_2. $p'=2$.

$p=3$.

Triple Conic.

124. The R_7 transforms into a non-singular plane C_4. $\bar{z}=\bar{w}=0$ meets the curve in 0 or 1 P_1'.

1. $d_4+C_2^3$; $(d_3+g_1)+C_2^3$.

Double Curve.

125. The R_7 has no P_3. It transforms by I into a C_6 meeting $\bar{x}=\bar{y}=0$ thrice and passing through each fundamental point. The scroll of bisecants is an R_7. $\bar{z}=\bar{w}=0$ meets the C_6 in 0 or 1 points besides the fundamental point.

1. $d_4+C_6^2$; $(d_3+g_1)+C_6^2$. The C_6 is of genus 3.

Directrix a Fivefold Line on the Surface.

$p=0$.

126. The R_7 transforms by I into a C_5 meeting $\bar{x}=\bar{y}=0$ thrice. The scroll of bisecants is an R_4. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2, 3 or 4 points.

127. When the R_7 has a g_2 it transforms into a C_5 with a P_2'. The scroll of bisecants is an R_3. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 or 3 P_1' or in the P_2' and 0, 1 or 2 P_1'.

128. From this configuration it is readily seen that the R_7 may have a contact directrix $(\delta_{2,2}+g_1)$ for such an R_7 is obtained when $\bar{z}=\bar{w}=0$ is taken

to coincide with the simple directrix of the R_3 of bisecants. Since both fundamental points lie on the R_3 of bisecants such an R_7 has $3g_2$.

129. When the R_7 has $2g_2$, it transforms into a C_5 with $2P'_2$. The scroll of bisecants is an R_2 or a K_2. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, or 2 P'_1 or in a P'_2 and 0 or 1 P'_1 or in $2P'_2$.

130. When the R_7 has $3g_2$ — and no $(\delta_{2,2}+g_1)$ — it transforms into a plane C_5 with a P'_3 on $\bar{x}=\bar{y}=0$ and $3P'_2$. $\bar{z}=\bar{w}=0$ meets the C_5 in 0 or 1 P'_1 or in a P'_2.

1. $d_5+C^2_5$; $(d_4+g_1)+C^2_5$; $(d_3+2g_1)+C^2_5$; $(d_2+3g_1)+C^2_5$; $(d_1+4g_1)+C^2_5$.

2. $d_5+g_2+C^2_4$; $(d_4+g_1)+g_2+C^2_4$; $(d_3+2g_1)+g_2+C^2_4$; $(d_2+3g_1)+g_2+C^2_4$; $(d_3+g_2)+C^2_4$; $(d_2+g_1+g_2)+C^2_4$; $(d_1+2g_1+g_2)+C^2_4$. The C_4 has a P'_1 on the g_2.

3. $d_5+2g_2+C^2_3$; $(d_4+g_1)+2g_2+C^2_3$; $(d_3+2g_1)+2g_2+C^2_3$; $(d_3+g_2)+g_2+C^2_3$; $(d_2+g_1+g_2)+g_2+C^2_3$; $(d_1+2g_2)+C^2_3$. The C_3 may be either plane or gauche. It has a P'_1 on each g_2.

4. $d_5+3g_2+C^2_2$; $(d_4+g_1)+3g_2+C^2_2$; $(d_3+g_2)+2g_2+C^2_2$. The C_2 has a P'_1 on each g_2.

5. $(\delta_{2,2}+g_1)+3g_2$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

$$p=1$$

131. The R_7 transforms by I into a C_5 of genus 1 having $3P'_1$ on $\bar{x}=\bar{y}=0$. The scroll of bisecants is an R_3. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1, 2 or 3 P'_1.

132. Taking the simple directrix of the R_3 of bisecants for $\bar{z}=\bar{w}=0$ we obtain a contact directrix. Since each fundamental point lies on the scroll of bisecants, the R_7 has $2g_2$.

133. When the R_7 has a g_2, the C_5 has a P'_2. The scroll of bisecants is an R_2 or a K_2. $\bar{z}=\bar{w}=0$ meets the C_5 in 0, 1 or 2 P'_1 or in the P'_2 and 0 or 1 P'_1.

133. When the R_7 has $2g_2$ — and no $(\delta_{2,2}+g_1)$ — it transforms into a plane C_5 with a P'_3 on $\bar{x}=\bar{y}=0$ and $2P'_2$. $\bar{z}=\bar{w}=0$ meets the C_5 in 0 or 1 P'_1 or in a P'_2.

1. $d_5+C^2_4$; $(d_4+g_1)+C^2_4$; $(d_3+2g_1)+C^2_4$; $(d_2+3g_1)+C^2_4$.

2. $d_5 + g_2 + C_3^2$; $(d_4 + g_1) + g_2 + C_3^2$; $(d_3 + 2g_1) + g_2 + C_3^2$; $(d_3 + g_2) + C_3^2$; $(d_2 + g_1 + g_2) + C_3^2$. The C_3 is either plane or gauche. It has a P_1' on the g_2.

3. $d_5 + 2g_2 + C_2^2$; $(d_4 + g_1) + 2g_2 + C_2^2$; $(d_3 + g_2) + g_2 + C_2^2$. The C_2 has a P_1' on each g_2.

4. $(\delta_{2,2} + g_1) + 2g_2$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

$$p = 2.$$

135. The R_7 transforms into a C_5 of genus two. The scroll of bisecants is an R_2 or a K_2. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0, 1 or 2 P_1'.

136. When the R_7 has a g_2 — and no $(\delta_{2,2} + g_1)$ —, it transforms into a a plane C_5 with a P_3' on $\bar{x} = \bar{y} = 0$ and a P_2'. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0 or 1 P_1' or in a P_2'.

137. When the R_7 has a contact directrix it has a g_2 and may therefore be transformed by II into a C_5 lying on K_2 which has its vertex at $\bar{x} = \bar{y} = \bar{z} = 0$ and has $\bar{x} = \bar{y} = 0$ for generator.

1. $d_5 + C_3^2$; $(d_4 + g_1) + C_3^2$; $(d_3 + 2g_1) + C_3^2$. The C_3 may be either plane or gauche.

2. $d_5 + g_2 + C_2^2$; $(d_4 + g_1) + g_2 + C_2^2$; $(d_3 + g_2) + C_2^2$. The C_2 has a P_1' on the g_2.

3. $(\delta_{2,2} + g_1) + g_2$. The two generators in an arbitrary plane through the directric intersect on the directrix.

$$p = 3.$$

138. When the R_7 does not have a contact directrix, it transforms into a plane C_5 with a P_3' on $\bar{x} = \bar{y} = 0$. $\bar{z} = \bar{w} = 0$ meets the C_5 in 0 or 1 P_1'.

139. When the R_7 has a contact directrix, it may be transformed by II into a C_6 having $\bar{x} = \bar{y} = \bar{z} = 0$ a P_2' with one branch tangent to $y = 0$ and lying on a K_2 which has $\bar{x} = \bar{y} = 0$ for generator.

1. $d_5 + C_2$; $(d_4 + g_1) + C_2$.

2. $(\delta_{2,2} + g_1)$. The two generators in an arbitrary plane through the directrix intersect on the directrix.

Directrix a Six-fold Line on the Surface.

140. The R_7 is necessarily unicursal. It transforms by I into a C_6 meeting $\bar{x}=\bar{y}=0$ five times. $\bar{z}=\bar{w}=0$ meets the C_6 in 0, 1, 2, 3, or 4 P_1'. If $\bar{z}=\bar{w}=0$ meets it in $5P_1'$ then R_7 belongs to a special linear congruence and has already been enumerated.

1. d_6; (d_5+g_1); (d_4+2g_1); (d_3+3g_1); (d_2+4g_1).

CORNELL UNIVERSITY, *July* 28, 1905.